IMU Modelling and Kinematics

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1 IMU Sensor Modelling and Kinematics

This document covers the basic IMU sensor model often used in robotics literature, which appears in many problems including visual-inertial, lidar-inertial, or radar-inertial navigation. The purpose of this document is to summarize

- The IMU kinematic model in both discrete-time and continuous-time,
- The Jacobians of the IMU process model in both discrete and continuous-time. These are useful for performing filtering with an IMU with the EKF, for example. The Jacobians are also important for performing observability analyses for inertial navigation problems.

To showcase the typical IMU sensor model, two frames are introduced - the inertial frame $\mathcal{F}_a$ and the frame that rotates with the IMU, $\mathcal{F}_b$. An unforced particle in $\mathcal{F}_a$ is denoted w and a reference position on the IMU is denoted point z. The direction cosine matrix (DCM) that relates the attitude of $\mathcal{F}_a$ to the attitude of $\mathcal{F}_b$ is denoted $\mathbf{C}_{ab} \in SO(3)$. The DCM relates physical vectors resolved in $\mathcal{F}_a$ to physical vectors resolved in $\mathcal{F}_b$ as $\mathbf{v}_a = \mathbf{C}_{ab}\mathbf{v}_b$. The IMU position and velocity resolved in the inertial frame are denoted $\mathbf{r}_a^{zw}$ and $\mathbf{v}_a^{zw/a}$, respectively.

1.1 IMU Sensor Modelling

This document focuses on a basic IMU sensor model that adds bias and noise to the true angular velocity and acceleration of the vehicle. The rate gyroscope measures

$$\mathbf{u}_b^g(t) = \boldsymbol{\omega}_b^{ba}(t) + \mathbf{b}_b^g(t) + \mathbf{w}_b^g(t), \tag{1}$$

where $\boldsymbol{\omega}_b^{ba}$ is the true angular velocity of the platform, $\mathbf{b}_b^g$ is the additive gyro bias resolved in the body frame, and $\mathbf{w}_b^g$ is white noise of the form $\mathbf{w}_b^g \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_c^g \delta(t-\tau))$, where $\mathbf{Q}_c^g$ is the power spectral density of the noise. Additionally, the gyro bias is modelled as a random walk $\dot{\mathbf{b}}_b^g = \mathbf{w}_b^{b_g}$, where $\mathbf{w}_b^{b_g} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_c^{b_g}\delta(t-\tau)\right)$. The accelerometer measurements are modelled to be of the form

$$\mathbf{u}_b^g(t) = \mathbf{C}_{ab}(t)^{\mathsf{T}}\left(\mathbf{a}_a^{zw/a/a}(t) + \mathbf{g}_a\right) + \mathbf{b}_b^a(t) + \mathbf{w}_b^a(t), \tag{2}$$

where $\mathbf{a}_a^{zw/a/a}(t)$ is the true acceleration of the IMU resolved in the inertial frame $\mathcal{F}_a$, and $\mathbf{g}_a$ is the gravity vector resolved in $\mathcal{F}_a$ Additionally, here $\mathbf{b}_b^a$ is the accelerometer bias which is also modelled as a random walk, such that $\dot{\mathbf{b}}_b^a = \mathbf{w}_b^{b_a}$, where $\mathbf{w}_b^{b_a} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_c^{b_a}\delta(t-\tau)\right)$.

1.2 Continuous-Time IMU Kinematics

The IMU navigation state consists of the IMU orientation, velocity, and position, written as $\mathcal{X}^{\text{nav}} = \left(\mathbf{C}_{ab}, \mathbf{v}_a^{zw/a}, \mathbf{r}_a^{zw}\right)$. The time evolution of the IMU navigation state can be written in terms of the true angular velocity and acceleration of the platform, as

$$\dot{\mathbf{C}}_{ab}(t) = \mathbf{C}_{ab}(t)\,\boldsymbol{\omega}_b^{ba}(t)^{\times}, \tag{3}$$

$$\dot{\mathbf{v}}_a^{zw/a}(t) = \mathbf{C}_{ab}(t)\,\mathbf{a}_b^{zw/a/a}(t) + \mathbf{g}_a, \tag{4}$$

$$\dot{\mathbf{r}}_a^{zw}(t) = \mathbf{v}_a^{zw/a}(t). \tag{5}$$

where $(\cdot)^{\times}$ is the skew-symmetric operator, a mapping from $\mathbb{R}^3$ to $\mathfrak{so}(3)$ such that $\mathbf{a}^{\times}\mathbf{b} = -\mathbf{b}^{\times}\mathbf{a}, \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Subbing in the IMU measurements rather than the true angular velocity and body frame acceleration leads to the continuous-time IMU kinematics given by

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}\left(\mathbf{u}_b^g - \mathbf{b}_b^g - \mathbf{w}_b^g\right)^{\times}, \tag{6}$$

$$\dot{\mathbf{v}}_a^{zw/a} = \mathbf{C}_{ab}\left(\mathbf{u}_b^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right) + \mathbf{g}_a, \tag{7}$$

$$\dot{\mathbf{r}}_a^{zw} = \mathbf{v}_a^{zw/a}, \tag{8}$$

$$\dot{\mathbf{b}}_b^g = \mathbf{w}_b^{b_g}, \tag{9}$$

$$\dot{\mathbf{b}}_b^a = \mathbf{w}_b^{b_a}. \tag{10}$$

Placing the attitude, velocity and position into an element of the matrix Lie group $SE_2(3)$ allows for a more compact representation of the above continuous-time IMU kinematics. The IMU navigation state can be placed into an element of $SE_2(3)$ as

$$\mathbf{T}_{ab} = \begin{bmatrix} \mathbf{C}_{ab} & \mathbf{v}_a^{zw/a} & \mathbf{r}_a^{zw} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \in SE_2(3), \tag{11}$$

Dropping the subscripts on $\mathbf{T}_{ab}$ for brevity, the identical continuous-time kinematics can be written in a more compact form as

$$\dot{\mathbf{T}} = \mathbf{G}\mathbf{T} + \mathbf{T}\left(\mathbf{U} - \mathbf{B}\right), \tag{12}$$

$$\dot{\mathbf{b}}_b = \mathbf{w}_b^b, \tag{13}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{g} & \mathbf{0} \\ \mathbf{0} & 0 & -1 \\ \mathbf{0} & 0 & 0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} \mathbf{u}_b^{g^{\times}} & \mathbf{u}_b^a & \mathbf{0} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_b^{g^{\times}} & \mathbf{b}_b^a & \mathbf{0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Note that here, the matrix $\mathbf{G}$ is always constant, the matrix $\mathbf{U}$ is a function of the IMU measurements, and the matrix $\mathbf{B}$ is a function of the IMU biases.

1.3 Discrete-Time IMU Kinematics

Implementing estimation algorithms involving an IMU requires a discretization of the continuous-time IMU kinematics. Several discretization schemes are possible, corresponding to different hypotheses on the robot motion between integration times. Some examples of common assumptions include

- The acceleration in the inertial frame, $\mathbf{a}_a^{zw/a/a}$ is roughly constant over the interval $t \in [i, j])$. This is the most common assumption used and can be found in, for example, [1], but is an assumption that is often be violated in practice. Nonetheless, a high enough IMU frequency can make this assumption reasonable.

- The IMU measurements $\mathbf{u}_b^g$ and $\mathbf{u}_b^a$ are constant over the integration interval. This discretization strategy can be found in [2], and lead to *exact* closed form expressions for the continuous-time kinematics.

- The *true local acceleration* $\mathbf{a}_b^{zw/a/a}$ is constant over the integration interval. The details of this assumption is presented in [3].

The subsequent section will explore the first two discretization schemes - constant inertial acceleration and constant IMU measurements.

1.3.1 IMU Discretization - Constant Inertial Acceleration

Denote the IMU attitude, velocity, and position at time $t = t_k$ as $\mathbf{C}_k = \mathbf{C}_{ab_k}$, $\mathbf{v}_k = \mathbf{v}_a^{z_k w}$, and $\mathbf{r}_k = \mathbf{r}_a^{z_k w}$. Under the assumption of constant inertial acceleration between two times, $t = t_{k-1}$ and $t = t_k$, the IMU kinematics can be discretized as

$$\mathbf{C}_k = \mathbf{C}_{k-1} \operatorname{Exp}\left(\left(\mathbf{u}_{k-1}^g - \mathbf{b}_{k-1}^g - \mathbf{w}_{k-1}^g\right) \Delta t\right) \tag{16}$$

$$\mathbf{v}_k = \mathbf{v}_{k-1} + \mathbf{C}_{k-1}\left(\mathbf{u}_{k-1}^a - \mathbf{b}_{k-1}^a - \mathbf{w}_{k-1}^a\right) \Delta t + \mathbf{g}_a \Delta t \tag{17}$$

$$\mathbf{r}_k = \mathbf{r}_{k-1} + \mathbf{v}_{k-1}\Delta t + \frac{1}{2}\left(\mathbf{C}_{k-1}\left(\mathbf{u}_{k-1}^g - \mathbf{b}_{k-1}^g - \mathbf{w}_{k-1}^g\right) + \mathbf{g}_a\right) \Delta t^2, \tag{18}$$

where $\Delta t = t_k - t_{k-1}$ is the time step between IMU measurements. This is the simplest discretization possible and

1.3.2 IMU Kinematics Discretization - Constant IMU Measurements

Rather than assume that the acceleration $\mathbf{a}_a^{zw/a/a}$ is constant, we can instead assume that the measurements $\mathbf{u}^g$ and $\mathbf{u}^a$ are constant and applying a zero-order hold on these measurements. Here, the use of the compact IMU kinematics, written as $\dot{\mathbf{T}} = \mathbf{G}\mathbf{T} + \mathbf{T}\left(\mathbf{U} - \mathbf{B}\right)$ makes this derivation significantly easier.

Assuming that the measurements are constant over the integration interval is equivalent to assuming that $\mathbf{U}$ is constant between a small integration interval Δt. With this assumption, the solution to this ODE with initial condition $\mathbf{T}_{k-1}$ is given by

$$\mathbf{T}_k = \exp\left(\Delta t \mathbf{G}\right) \mathbf{T}_{k-1}\left(\Delta t \mathbf{U}\right) \tag{19}$$

$$= \mathbf{G}_{k-1}\mathbf{T}_{k-1}\mathbf{U}_{k-1}, \tag{20}$$

where the matrices $\mathbf{G}_{k-1}$ and $\mathbf{U}_{k-1}$ are computed in closed form by a direct series expansion of the matrix exponential and are given by

$$\mathbf{G}_{k-1} = \begin{bmatrix} \mathbf{1} & \Delta t \mathbf{g}_a & -\left(\Delta t^2/2\right)\mathbf{g}_a \\ 0 & 1 & -\Delta t \\ 0 & 0 & 1 \end{bmatrix} \tag{21}$$

$$\mathbf{U}_{k-1} = \begin{bmatrix} \exp\left(\Delta t \boldsymbol{\omega}_{k-1}^{\wedge}\right) & \Delta t \mathbf{J}_\ell\left(\Delta t \boldsymbol{\omega}\right)\mathbf{a}_{k-1} & \left(\Delta t^2/2\right)\mathbf{N}\left(\Delta t \boldsymbol{\omega}_{k-1}\right)\mathbf{a}_{k-1} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

where here, $\boldsymbol{\omega}_{k-1} = \mathbf{u}_{k-1}^g - \mathbf{b}_{k-1}^g - \mathbf{w}_{k-1}^g$ and $\mathbf{a}_{k-1} = \mathbf{u}_{k-1}^a - \mathbf{b}_{k-1}^a - \mathbf{w}_{k-1}^a$, and additionally, the expression $\mathbf{N}\left(\cdot\right)$ is given by

$$\mathbf{N}\left(\boldsymbol{\phi}\right) = \mathbf{a}\mathbf{a}^{\mathsf{T}} + 2\left(\frac{1}{\phi} - \frac{\sin\phi}{\phi^2}\right)\mathbf{a}^{\times} + 2\frac{\cos\phi - 1}{\phi^2}\mathbf{a}^{\times}\mathbf{a}^{\times}, \tag{23}$$

where $\mathbf{a} = \boldsymbol{\phi}/\left\|\boldsymbol{\phi}\right\|$ and $\phi = \left\|\boldsymbol{\phi}\right\|$.

Note that additionally, $\mathbf{U}_{k-1}$ is *not* an element of $SE_2(3)$. However, it can be decomposed into a product of an element of $SE_2(3)$ and another matrix, written as

$$\mathbf{U}_{k-1} = \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \Delta t \\ \mathbf{0} & 0 & 1 \end{bmatrix}}_{\boldsymbol{\Delta}} \underbrace{\begin{bmatrix} \exp\left(\Delta t \boldsymbol{\omega}^{\wedge}\right) & \Delta t \mathbf{J}_\ell\left(\Delta t \boldsymbol{\omega}\right)\mathbf{a} & \left(\Delta t^2/2\right)\mathbf{N}\left(\Delta t \boldsymbol{\omega}\right)\mathbf{a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{U}_{k-1}^{\text{nav}}}, \tag{24}$$

where $\mathbf{U}_{k-1}$ is an element of $SE_2(3)$. Note that these discrete-time kinematics match those found in [2, p.101], but the use of the compact form makes the derivation simpler.

1.4 State, Noise, and Error Definitions

The state for estimation in inertial navigation problems is typically the IMU orientation, velocity, and position, collectively refered to as the *navigation* state, as well as the IMU biases. Stacking the IMU biases as

$$\mathbf{b}_b = \begin{bmatrix} \mathbf{b}_b^g \\ \mathbf{b}_b^a \end{bmatrix} \in \mathbb{R}^6, \tag{25}$$

and placing the IMU attitude, velocity, and position into an element of $SE_2(3)$ denoted $\mathbf{T}_{ab}$, the full IMU state is then defined as

$$\mathcal{X} = \left(\mathbf{T}_{ab}, \mathbf{b}_b\right) \in SE_2(3) \times \mathbb{R}^6. \tag{26}$$

This is a composite Lie group, where the group composition operation can be defined as

$$\mathcal{X}_1 \circ \mathcal{X}_2 = \left(\mathbf{T}_{ab_1}\mathbf{T}_{ab_2}, \mathbf{b}_{b_1} + \mathbf{b}_{b_2}\right). \tag{27}$$

All other important operations are defined analogously, where each individual operation acts separately on the IMU navigation state and the IMU biases.

This definition of the IMU state allows for the definition of the $\oplus$ operator given by

$$\mathcal{X} \oplus \delta\boldsymbol{\xi} = \left(\mathbf{T}_{ab} \oplus \delta\boldsymbol{\xi}^{\text{nav}}, \mathbf{b}_b + \delta\boldsymbol{\xi}^b\right), \tag{28}$$

$$\delta\boldsymbol{\xi} = \begin{bmatrix} \delta\boldsymbol{\xi}^{\text{nav}} \\ \delta\boldsymbol{\xi}^b \end{bmatrix} \in \mathbb{R}^{15}, \tag{29}$$

where $\delta\boldsymbol{\xi}^{\text{nav}} \in \mathbb{R}^9$ is the perturbation to the navigation state and $\delta\boldsymbol{\xi}^b \in \mathbb{R}^6$ is the perturbation to the IMU biases. Both the left and right definitions of the $\oplus$ operator can be used in (28). Additionally, the $\ominus$ operator for the state is defined analogoously as

$$\mathcal{X}_1 \ominus \mathcal{X}_2 = \begin{bmatrix} \mathbf{T}_{ab_1} \ominus \mathbf{T}_{ab_2} \\ \mathbf{b}_{b_1} - \mathbf{b}_{b_2}, \end{bmatrix} \in \mathbb{R}^{15}, \tag{30}$$

where a left or right definition of the $\ominus$ operator can be used for the navigation state.

Finally, the final definition needed before deriving linearized dynamics for the IMU process model is the definition of the noises. The IMU white noises and random walk noises can be stacked to define a noise matrix as

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_b^{g^\mathsf{T}} & \mathbf{w}_b^{a^\mathsf{T}} & \mathbf{w}_b^{b_g^\mathsf{T}} & \mathbf{w}_b^{b_a^\mathsf{T}} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{12}, \tag{31}$$

with noise PSD matrix given by

$$\mathbf{Q}_c = \operatorname{diag}\left(\mathbf{Q}_c^g, \mathbf{Q}_c^a, \mathbf{Q}_c^{b_g}, \mathbf{Q}_c^{b_a}\right). \tag{32}$$

With these state, noise, and error definitions, the continuous-time process model can now be linearized.

1.5 Continuous-Time Process Model Linearization

The Jacobians of the IMU process model with the state are often needed for estimation (i.e., in an EKF), or for analysis purposes (to perform an observability analysis, for example). This section will outline the forms of the continuous-time process model Jacobians for both the left and right perturbation of the IMU navigation state.

The continuous-time process model is written as

$$\dot{\mathcal{X}}(t) = \left(\dot{\mathbf{T}}(t), \dot{\mathbf{b}}(t)\right) = f\left(\mathcal{X}(t), \mathbf{u}(t), \mathbf{w}(t)\right), \tag{33}$$

where the function $f\left(\mathcal{X}(t), \mathbf{u}(t), \mathbf{w}(t)\right)$ can be written as

$$\left(\dot{\mathbf{T}}, \dot{\mathbf{b}}\right) = \left(f_1\left(\mathcal{X}(t), \mathbf{u}(t), \mathbf{t}\right), f_2\left(\mathcal{X}(t), \mathbf{u}(t), \mathbf{w}(t)\right)\right). \tag{34}$$

The function $f_1\left(\mathcal{X}, \mathbf{u}, \mathbf{w}\right)$ is the process model for the navigation state, and the function $f_2\left(\mathcal{X}, \mathbf{u}, \mathbf{w}\right)$ is the process model for the IMU biases, where these are written as

$$\dot{\mathbf{T}} = \mathbf{GT} + \mathbf{TU}, \tag{35}$$

$$\dot{\mathbf{b}} = \mathbf{w}^b. \tag{36}$$

Linearizing the process-model in continuous-time yields an equation of the form

$$\delta\dot{\boldsymbol{\xi}} \approx \mathbf{F}\delta\boldsymbol{\xi} + \mathbf{L}\delta\mathbf{w}, \tag{37}$$

where $\mathbf{F}$ is the continuous-time Jacobian of the process model with respect to $\mathcal{X}$, and $\mathbf{L}$ is the continuous-time Jacobian of the process model with respect to the noise. Formally, these are written as

$$\mathbf{F}_c = \frac{Df(\mathcal{X}, \mathbf{u}, \mathbf{w})}{D\mathcal{X}}, \tag{38}$$

$$\mathbf{L}_c = \frac{Df(\mathcal{X}, \mathbf{u}, \mathbf{w})}{D\mathbf{w}}, \tag{39}$$

where the Lie group definition of a Jacobian is used. The exact expressions for the Jacobians $\mathbf{F}_c$ and $\mathbf{L}_c$ depend on whether a left or right definition of the $\oplus$ operator is used. Starting with a right definition of the $\oplus$ operator, these Jacobians are given as

$$\mathbf{F}_c = \begin{bmatrix} -\left(\mathbf{u}^g - \mathbf{b}^g\right)^\times & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ -\left(\mathbf{u}^a - \mathbf{b}^a\right) & -\left(\mathbf{u}^g - \mathbf{b}^g\right)^\times & \mathbf{0} & \mathbf{0} & -\mathbf{1} \\ \mathbf{0} & \mathbf{1} & -\left(\mathbf{u}^g - \mathbf{b}^g\right)^\times & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{L}_c = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{40}$$

For a left definition of the $\oplus$ operator, the continuous-time process model Jacobians are given by

$$\mathbf{F}_c = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{ab} & \mathbf{0} \\ \mathbf{g}^\times & \mathbf{0} & \mathbf{0} & -\mathbf{v}_a^{zw/a^\times}\mathbf{C}_{ab} & -\mathbf{C}_{ab} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{r}_a^{zw^\times}\mathbf{C}_{ab} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{L}_c = \begin{bmatrix} \mathbf{C}_{ab} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{v}_a^{zw/a^\times}\mathbf{C}_{ab} & \mathbf{C}_{ab} & \mathbf{0} & \mathbf{0} \\ \mathbf{r}_a^{zw^\times}\mathbf{C}_{ab} & \mathbf{0} & \mathbf{C}_{ab} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{41}$$

Note that the continuous-time matrix $\mathbf{F}_c$ depends on the input and biases for a right perturbation definition, whereas it depends on the navigation state for the left perturbation definition. The full derivations for these Jacobians can be found in Appendix A.

To implement a state estimation algorithm like an EKF, the continuous-time linear dynamics can be discretized using any method of choice, to yield the discrete-time linear system given by

$$\delta\boldsymbol{\xi}_k \approx \mathbf{F}_{k-1}\delta\boldsymbol{\xi}_{k-1} + \delta\mathbf{w}_{k-1}, \tag{42}$$

where $\delta\mathbf{w}_{k-1} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_{k-1}\right)$.

References

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A Continuous-Time IMU Kinematics Jacobian Derivations

In this section, the Jacobians of the continuous-time IMU kinematics will be derived for both a left and right perturbation of the navigation state. Neglecting subscripts from brevity, recall that the continuous-time IMU kinematics are given by

$$\dot{\mathbf{C}} = \mathbf{C}\left(\mathbf{u}^g - \mathbf{b}^g - \mathbf{w}^g\right), \tag{43}$$
$$\dot{\mathbf{v}} = \mathbf{C}\left(\mathbf{u}^a - \mathbf{b}^a - \mathbf{w}^a\right) + \mathbf{g}, \tag{44}$$
$$\dot{\mathbf{r}} = \mathbf{v}, \tag{45}$$
$$\dot{\mathbf{b}}^g = \mathbf{w}^b, \tag{46}$$
$$\dot{\mathbf{b}}^a = \mathbf{w}^a. \tag{47}$$

The next sections will derive the error dynamics of this continuous-time process model for both a left and right perturbation of the state.

A.1 Left Perturbation Derivation

To start, consider the state definition $\mathcal{X} = (\mathbf{T}, \mathbf{b}) \in SE_2(3) \times \mathbb{R}^6$, and the perturbation given by

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \delta\boldsymbol{\xi} = \left(\bar{\mathbf{T}} \oplus \delta\boldsymbol{\xi}^{\text{nav}}, \bar{\mathbf{b}} + \delta\boldsymbol{\xi}^b\right). \tag{48}$$

Expanding the navigation state perturbation yields

$$\mathbf{T} = \operatorname{Exp}\left(\delta\boldsymbol{\xi}^{\text{nav}}\right)\bar{\mathbf{T}}, \tag{49}$$
$$\begin{bmatrix} \mathbf{C} & \mathbf{v} & \mathbf{r} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta\mathbf{C} & \delta\mathbf{v} & \delta\mathbf{r} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{C}} & \bar{\mathbf{v}} & \bar{\mathbf{r}} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix}, \tag{50}$$
$$= \begin{bmatrix} \delta\mathbf{C}\bar{\mathbf{C}} & \delta\mathbf{C}\bar{\mathbf{v}} + \delta\mathbf{v} & \delta\mathbf{C}\bar{\mathbf{r}} + \delta\mathbf{r} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix}. \tag{51}$$

Rearranging yields the individual error definitions for the navigation state as

$$\delta\mathbf{C} = \mathbf{C}\bar{\mathbf{C}}^\mathsf{T}, \tag{52}$$
$$\delta\mathbf{v} = \mathbf{v} - \delta\mathbf{C}\bar{\mathbf{v}}, \tag{53}$$
$$\delta\mathbf{r} = \mathbf{r} - \delta\mathbf{C}\bar{\mathbf{r}}. \tag{54}$$

The error dynamics of the continuous-time IMU kinematics can now be derived by taking the time derivative of the error definitions. Starting with the attitude yields

$$\delta\dot{\mathbf{C}} = \dot{\mathbf{C}}\bar{\mathbf{C}}^\mathsf{T} + \mathbf{C}\dot{\bar{\mathbf{C}}}^\mathsf{T} \tag{55}$$
$$= \mathbf{C}\left(\mathbf{u}^g - \mathbf{b}^g - \mathbf{w}^g\right)^\times \bar{\mathbf{C}}^\mathsf{T} - \mathbf{C}\left(\mathbf{u}^g - \bar{\mathbf{b}}^g\right)^\times \bar{\mathbf{C}}^\mathsf{T}, \tag{56}$$
$$= \mathbf{C}\left(-\mathbf{b}^g + \bar{\mathbf{b}}^g - \mathbf{w}^g\right)^\times \bar{\mathbf{C}}^\mathsf{T}, \tag{57}$$
$$= \delta\mathbf{C}\bar{\mathbf{C}}\left(-\delta\mathbf{b}^g - \mathbf{w}^g\right)^\times \bar{\mathbf{C}}^\mathsf{T}, \tag{58}$$
$$= \delta\mathbf{C}\left(\bar{\mathbf{C}}\left(-\delta\mathbf{b}^g - \mathbf{w}^g\right)\right)^\times \tag{59}$$

Linearizing by letting $\delta\mathbf{C} \approx \mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}$ and neglecting products of small terms yields

$$\delta\dot{\boldsymbol{\xi}}^{\phi\times} \approx \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}\right)\left(\bar{\mathbf{C}}\left(-\delta\boldsymbol{\xi}^{b_g} - \mathbf{w}^g\right)\right)^\times \tag{60}$$

$$\approx -\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_g} - \bar{\mathbf{C}}\delta\mathbf{w}^g. \tag{61}$$

Next, consider the velocity error dynamics given by

$$\delta\dot{\mathbf{v}} = \dot{\mathbf{v}} - \delta\dot{\mathbf{C}}\bar{\mathbf{v}} - \delta\mathbf{C}\dot{\bar{\mathbf{v}}} \tag{62}$$

$$= \mathbf{C}\left(\mathbf{u}^a - \mathbf{b}^a - \mathbf{w}^a\right) + \mathbf{g} - \delta\dot{\mathbf{C}}\bar{\mathbf{v}} - \delta\mathbf{C}\left(\bar{\mathbf{C}}\left(\mathbf{u}^a - \bar{\mathbf{b}}^a\right) + \mathbf{g}\right) \tag{63}$$

$$= \delta\mathbf{C}\bar{\mathbf{C}}\left(\mathbf{u}^a - \bar{\mathbf{b}}^a - \delta\mathbf{b}^a - \mathbf{w}^a\right) + \mathbf{g} - \delta\dot{\mathbf{C}}\bar{\mathbf{v}} - \delta\mathbf{C}\left(\bar{\mathbf{C}}\left(\mathbf{u}^a - \bar{\mathbf{b}}^a\right) + \mathbf{g}\right), \tag{64}$$

$$= \delta\mathbf{C}\bar{\mathbf{C}}\left(\delta\mathbf{b}^a - \mathbf{w}^a\right) + \mathbf{g} - \delta\dot{\mathbf{C}}\bar{\mathbf{v}} - \delta\mathbf{C}\mathbf{g}_a. \tag{65}$$

Linearizing by letting $\delta\mathbf{C} \approx \mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}$, and $\delta\mathbf{v} \approx \delta\boldsymbol{\xi}^v$, and $\delta\dot{\mathbf{C}} \approx \left(-\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_g} - \bar{\mathbf{C}}\delta\mathbf{w}^g\right)^\times$ yields

$$\delta\dot{\boldsymbol{\xi}}^v \approx \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}\right)\bar{\mathbf{C}}\left(-\delta\boldsymbol{\xi}^{b_a} - \delta\mathbf{w}^a\right) + \mathbf{g} - \left(-\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_g} - \bar{\mathbf{C}}\delta\mathbf{w}^g\right)^\times\bar{\mathbf{v}} - \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}\right)\mathbf{g}, \tag{66}$$

$$= -\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_a} - \bar{\mathbf{C}}\delta\mathbf{w}^a - \bar{\mathbf{v}}^\times\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_g} - \bar{\mathbf{v}}^\times\bar{\mathbf{C}}\delta\mathbf{w}^g + \mathbf{g}^\times\delta\boldsymbol{\xi}^\phi. \tag{67}$$

Next, consider error dynamics $\delta\dot{\mathbf{r}}$ given by

$$\delta\dot{\mathbf{r}} = \dot{\mathbf{r}} - \delta\dot{\mathbf{C}}\bar{\mathbf{r}} - \delta\mathbf{C}\dot{\bar{\mathbf{r}}}, \tag{68}$$

$$= \mathbf{v} - \delta\dot{\mathbf{C}}\bar{\mathbf{r}} - \delta\mathbf{C}\bar{\mathbf{v}}, \tag{69}$$

$$= \delta\mathbf{C}\bar{\mathbf{v}} + \delta\mathbf{v} - \delta\dot{\mathbf{C}}\bar{\mathbf{r}} - \delta\mathbf{C}\bar{\mathbf{v}} \tag{70}$$

$$= \delta\mathbf{v} - \delta\dot{\mathbf{C}}\bar{\mathbf{r}} \tag{71}$$

Linearizing by letting $\delta\mathbf{C} \approx \mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}$, $\delta\mathbf{v} \approx \delta\boldsymbol{\xi}^v$, and $\delta\dot{\mathbf{C}} \approx \left(-\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_g} - \bar{\mathbf{C}}\delta\mathbf{w}^g\right)^\times$ yields

$$\delta\dot{\boldsymbol{\xi}}^r \approx \delta\boldsymbol{\xi}^v - \left(-\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_g} - \bar{\mathbf{C}}\delta\mathbf{w}^g\right)^\times\bar{\mathbf{r}}, \tag{72}$$

$$= \delta\boldsymbol{\xi}^v - \bar{\mathbf{r}}^\times\bar{\mathbf{C}}\delta\boldsymbol{\xi}^{b_g} - \bar{\mathbf{r}}^\times\bar{\mathbf{C}}\delta\mathbf{w}^g. \tag{73}$$

Finally, linearizing the bias error dynamics is given by

$$\delta\dot{\mathbf{b}} = \dot{\mathbf{b}} - \dot{\bar{\mathbf{b}}} = \mathbf{w}, \tag{74}$$

$$\delta\dot{\mathbf{b}} \approx \delta\mathbf{w}. \tag{75}$$

Hence, the continous-time process model Jacobians for a left perturbation of the navigation state is given by

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{g}^\times & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{v}}^\times\bar{\mathbf{C}} & -\bar{\mathbf{C}} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & -\bar{\mathbf{r}}^\times\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{76}$$

$$\mathbf{L} = \begin{bmatrix} \bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{v}}^\times\bar{\mathbf{C}} & \bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{r}}^\times\bar{\mathbf{C}} & \mathbf{0} & \bar{\mathbf{0}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{77}$$

A.2 Right Perturbation

Next, the same state definition $\mathcal{X} = (\mathbf{T}, \mathbf{b}) \in SE_2(3) \times \mathbb{R}^6$ will be considered, but the perturbation on the navigation state will be a right perturbation given by

$$\mathbf{T} = \bar{\mathbf{T}} \, \mathrm{Exp}\left(\delta \boldsymbol{\xi}^{\mathrm{nav}}\right), \tag{78}$$

$$\begin{bmatrix} \mathbf{C} & \mathbf{v} & \mathbf{r} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{C}} & \bar{\mathbf{v}} & \bar{\mathbf{r}} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\mathbf{C} & \delta\mathbf{v} & \delta\mathbf{r} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \tag{79}$$

$$= \begin{bmatrix} \bar{\mathbf{C}}\delta\mathbf{C} & \bar{\mathbf{C}}\delta\mathbf{v} + \bar{\mathbf{v}} & \bar{\mathbf{C}}\delta\mathbf{r} + \bar{\mathbf{r}} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix}. \tag{80}$$

This leads to the individual error definitions given by

$$\delta\mathbf{C} = \bar{\mathbf{C}}^{\mathsf{T}}\mathbf{C}, \tag{81}$$

$$\delta\mathbf{v} = \bar{\mathbf{C}}^{\mathsf{T}}\left(\mathbf{v} - \bar{\mathbf{v}}\right), \tag{82}$$

$$\delta\mathbf{r} = \bar{\mathbf{C}}^{\mathsf{T}}\left(\mathbf{r} - \bar{\mathbf{r}}\right). \tag{83}$$

Starting with the attitude error dynamics yields

$$\delta\dot{\mathbf{C}} = \dot{\bar{\mathbf{C}}}^{\mathsf{T}}\mathbf{C} + \bar{\mathbf{C}}^{\mathsf{T}}\dot{\mathbf{C}} \tag{84}$$

$$= -\left(\mathbf{u}^g - \bar{\mathbf{b}}^g\right)^{\times} \bar{\mathbf{C}}^{\mathsf{T}}\mathbf{C} + \bar{\mathbf{C}}^{\mathsf{T}}\mathbf{C}\left(\mathbf{u}^g - \mathbf{b}^g - \mathbf{w}^g\right)^{\times}, \tag{85}$$

$$= -\left(\mathbf{u}^g - \bar{\mathbf{b}}^g\right)^{\times} \delta\mathbf{C} + \delta\mathbf{C}\left(\mathbf{u}^g - \bar{\mathbf{b}}^g - \delta\mathbf{b}^g - \mathbf{w}^g\right)^{\times}. \tag{86}$$

Linearizing by letting $\delta\mathbf{C} \approx \mathbf{1} + \delta\boldsymbol{\xi}^{\phi^{\times}}$ and $\delta\mathbf{b}^g \approx \delta\boldsymbol{\xi}^{b_g}$ yields

$$\delta\dot{\boldsymbol{\xi}}^{\phi^{\times}} \approx -\left(\mathbf{u}^g - \bar{\mathbf{b}}^g\right)^{\times}\left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi^{\times}}\right) + \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi^{\times}}\right)\left(\mathbf{u}^g - \bar{\mathbf{b}}^g - \delta\boldsymbol{\xi}^{b_g} - \delta\mathbf{w}^g\right)^{\times}, \tag{87}$$

$$\delta\dot{\boldsymbol{\xi}}^{\phi} \approx -\left(\mathbf{u}^g - \bar{\mathbf{b}}^g\right)^{\times}\delta\boldsymbol{\xi}^{\phi^{\times}} - \delta\boldsymbol{\xi}^{b_g} - \delta\mathbf{w}^g. \tag{88}$$

Next, consider the error dynamics $\delta\dot{\mathbf{v}}$ given by

$$\delta\dot{\mathbf{v}} = \dot{\mathbf{C}}^{\mathsf{T}}\left(\mathbf{v} - \bar{\mathbf{v}}\right) + \bar{\mathbf{C}}^{\mathsf{T}}\left(\dot{\mathbf{v}} - \dot{\bar{\mathbf{v}}}\right). \tag{89}$$

Subbing in the appropriate perturbations and linearizing yields

$$\delta\boldsymbol{\xi}^{v} \approx -\left(\mathbf{u}^a - \bar{\mathbf{b}}^a\right)^{\times}\delta\boldsymbol{\xi}^{\phi} - \left(\mathbf{u}^g - \mathbf{b}^g\right)^{\times}\delta\boldsymbol{\xi}^{v} - \delta\boldsymbol{\xi}^{b_a} - \delta\mathbf{w}^a. \tag{90}$$

Next, consider the error dynamics $\delta\dot{\mathbf{r}}$ given by

$$\delta\dot{\mathbf{r}} = \dot{\bar{\mathbf{C}}}^{\mathsf{T}}\left(\mathbf{r} - \bar{\mathbf{r}}\right) + \bar{\mathbf{C}}^{\mathsf{T}}\left(\dot{\mathbf{r}} - \dot{\bar{\mathbf{r}}}\right), \tag{91}$$

$$= -\left(\mathbf{u}^g - \bar{\mathbf{b}}^g\right)^{\times}\delta\mathbf{r} + \delta\mathbf{v} \tag{92}$$

$$\approx -\left(\mathbf{u}^g - \bar{\mathbf{b}}^g\right)^{\times}\delta\boldsymbol{\xi}^{r} + \delta\boldsymbol{\xi}^{v}. \tag{93}$$

The bias error dynamics are identical to the previous derivation. All together, utilizing a right perturbation for the navigation state yields the continuous-time Jacobians given by

$$\mathbf{F}_c = \begin{bmatrix} -\left(\mathbf{u}^g - \mathbf{b}^g\right)^\times & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ -\left(\mathbf{u}^a - \mathbf{b}^a\right) & -\left(\mathbf{u}^g - \mathbf{b}^g\right)^\times & \mathbf{0} & \mathbf{0} & -\mathbf{1} \\ \mathbf{0} & \mathbf{1} & -\left(\mathbf{u}^g - \mathbf{b}^g\right)^\times & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{L}_c \quad = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{94}$$

A.3 Discrete-Time State Transition Matrix

This section will derive the discrete-time state transition matrix $\dot{\mathbf{\Phi}}_{k,1}$, specifically for a left perturbation of the state. Recall that the state transition matrix $\mathbf{\Phi}_{k,1}$ satisfies the following matrix differential equation [4]

$$\dot{\mathbf{\Phi}}_{k,1} = \mathbf{F}_c\left(t\right)\mathbf{\Phi}_{k,1}, \quad \mathbf{\Phi}_{1,1} = \mathbf{1}, \tag{95}$$

where utilizing a left perturbation of the navigation state, $\mathbf{F}_c$ has the following structure.

$$\mathbf{F}_c(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{C} & \mathbf{0} \\ \mathbf{g}^\times & \mathbf{0} & \mathbf{0} & -\mathbf{v}^\times\mathbf{C} & -\mathbf{C} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{r}^\times\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{96}$$

(97)

where $\mathbf{C}(t), \mathbf{v}(t)$ and $\mathbf{r}(t)$ are nominal time-varying values for the attitude, velocity, and position. Examining the block elements of $\dot{\mathbf{\Phi}}_{k,1}$ allows us to determine an analytical solution for the state transition matrix. Consider the multiplication given by

$$\dot{\mathbf{\Phi}}_{k,1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\mathbf{r}^\times\mathbf{C} & \mathbf{0} \\ \mathbf{g}^\times & \mathbf{0} & \mathbf{0} & -\mathbf{v}^\times\mathbf{C} & -\mathbf{C} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{k,1}^{11} & \mathbf{\Phi}_{k,1}^{12} & \mathbf{\Phi}_{k,1}^{13} & \mathbf{\Phi}_{k,1}^{14} & \mathbf{\Phi}_{k,1}^{15} \\ \mathbf{\Phi}_{k,1}^{21} & \mathbf{\Phi}_{k,1}^{22} & \mathbf{\Phi}_{k,1}^{23} & \mathbf{\Phi}_{k,1}^{24} & \mathbf{\Phi}_{k,1}^{25} \\ \mathbf{\Phi}_{k,1}^{31} & \mathbf{\Phi}_{k,1}^{32} & \mathbf{\Phi}_{k,1}^{33} & \mathbf{\Phi}_{k,1}^{34} & \mathbf{\Phi}_{k,1}^{35} \\ \mathbf{\Phi}_{k,1}^{41} & \mathbf{\Phi}_{k,1}^{42} & \mathbf{\Phi}_{k,1}^{43} & \mathbf{\Phi}_{k,1}^{44} & \mathbf{\Phi}_{k,1}^{45} \\ \mathbf{\Phi}_{k,1}^{51} & \mathbf{\Phi}_{k,1}^{52} & \mathbf{\Phi}_{k,1}^{53} & \mathbf{\Phi}_{k,1}^{54} & \mathbf{\Phi}_{k,1}^{55} \end{bmatrix} \tag{98}$$

Since the last two rows of the process model Jacobian are zero,

$$\dot{\mathbf{\Phi}}_{k,1}^{i\star} = \mathbf{0}, \quad i = 4, 5. \tag{99}$$

Utilizing the initial condition $\mathbf{\Phi}_{k,1} = \mathbf{1}$, $\mathbf{\Phi}_{k,1}^{i,i} = \mathbf{1}$ if $i = 4, 5$ and $\mathbf{0}$ otherwise. Thus, the state transition matrix can now be written as

$$\dot{\mathbf{\Phi}}_{k,1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\mathbf{r}^\times\mathbf{C} & \mathbf{0} \\ \mathbf{g}^\times & \mathbf{0} & \mathbf{0} & -\mathbf{v}^\times\mathbf{C} & -\mathbf{C} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{k,1}^{11} & \mathbf{\Phi}_{k,1}^{12} & \mathbf{\Phi}_{k,1}^{13} & \mathbf{\Phi}_{k,1}^{14} & \mathbf{\Phi}_{k,1}^{15} \\ \mathbf{\Phi}_{k,1}^{21} & \mathbf{\Phi}_{k,1}^{22} & \mathbf{\Phi}_{k,1}^{23} & \mathbf{\Phi}_{k,1}^{24} & \mathbf{\Phi}_{k,1}^{25} \\ \mathbf{\Phi}_{k,1}^{31} & \mathbf{\Phi}_{k,1}^{32} & \mathbf{\Phi}_{k,1}^{33} & \mathbf{\Phi}_{k,1}^{34} & \mathbf{\Phi}_{k,1}^{35} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{100}$$

Following a similar approach, expanding the first block row of the state transition matrix yields

$$\dot{\mathbf{\Phi}}_{k,1}^{11} = \mathbf{0}, \tag{101}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{14} = -\mathbf{C}, \tag{102}$$

Utilizing the initial condition $\mathbf{\Phi}_{k,1}^{11} = \mathbf{1}$, we have

$$\mathbf{\Phi}_{k,1}^{11} = \mathbf{1}, \tag{103}$$

$$\mathbf{\Phi}_{k,1}^{14} = -\int_{t_1}^{t_k} \mathbf{C}(\tau)\mathrm{d}\tau. \tag{104}$$

Next, consider expanding the expressions for the second block row of the state transition matrix as

$$\dot{\mathbf{\Phi}}_{k,1}^{21} = \mathbf{g}_a^{\times}, \tag{105}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{22} = \mathbf{0}, \tag{106}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{24} = \mathbf{g}_a^{\times}\mathbf{\Phi}_{k,1}^{14} - \mathbf{v}^{\times}\mathbf{C}, \tag{107}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{25} = -\mathbf{C}. \tag{108}$$

Integrating these from their initial conditions yields

$$\mathbf{\Phi}_{k,1}^{21} = \Delta t \mathbf{g}_a^{\times}, \tag{109}$$

$$\mathbf{\Phi}_{k,1}^{22} = \mathbf{1}, \tag{110}$$

$$\mathbf{\Phi}_{k,1}^{24} = -\int_{t_1}^{t_k} \left(\mathbf{v}\left(\tau\right)^{\times} \mathbf{C}\left(\tau\right) - \mathbf{g}_a^{\times} \int_{t_1}^{\tau} \mathbf{C}(s)\mathrm{d}s \right) \mathrm{d}\tau, \tag{111}$$

$$\mathbf{\Phi}_{k,1}^{25} = -\int_{t_1}^{t_k} \mathbf{C}(\tau)\mathrm{d}\tau, \tag{112}$$

where $\Delta t = t_k - t_1$. Next, expanding the expressions for the third block row of the state transition matrix yields

$$\dot{\mathbf{\Phi}}_{k,1}^{31} = \mathbf{\Phi}_{k,1}^{21}, \tag{113}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{32} = \mathbf{\Phi}_{k,1}^{22}, \tag{114}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{33} = \mathbf{0}, \tag{115}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{34} = \mathbf{\Phi}_{k,1}^{24} - \mathbf{r}^{\times}\mathbf{C}, \tag{116}$$

$$\dot{\mathbf{\Phi}}_{k,1}^{35} = \mathbf{\Phi}_{k,1}^{25}. \tag{117}$$

Integrating these from their initial conditions yields

$$\mathbf{\Phi}_{k,1}^{31} = \frac{1}{2}\mathbf{g}_a^{\times}\Delta t, \tag{118}$$

$$\mathbf{\Phi}_{k,1}^{32} = \Delta t\mathbf{1}, \tag{119}$$

$$\mathbf{\Phi}_{k,1}^{33} = \mathbf{1}, \tag{120}$$

$$\mathbf{\Phi}_{k,1}^{34} = -\int_{t_1}^{t_k} \left(\mathbf{r}\left(\theta\right)^{\times}\mathbf{C}(\theta)\right) \tag{121}$$

$$+ \left[\int_{t_1}^{\theta}\left(\mathbf{v}(\tau)^{\times}\mathbf{C}(\tau) + \mathbf{g}_a^{\times}\int_{t_1}^{\tau}\mathbf{C}(s)\mathrm{d}s\right)\mathrm{d}\tau\right]\mathrm{d}\theta, \tag{122}$$

$$\mathbf{\Phi}_{k,1}^{35} = -\int_{t_1}^{t_k}\left(\int_{t_1}^{\tau}\left(\mathbf{C}(s)\mathrm{d}s\right)\right)\mathrm{d}\tau, \tag{123}$$

$$\tag{124}$$

In summary, putting all of these components together, the discrete-time state transition matrix has the following structure.

$$\mathbf{\Phi}_{k,1} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{k,1}^{14} & \mathbf{0} & \mathbf{0} \\ \mathbf{g}_a^{\times}\Delta t & \mathbf{1} & \mathbf{0} & \mathbf{\Phi}_{k,1}^{24} & \mathbf{\Phi}_{k,1}^{25} & \mathbf{0} \\ \frac{1}{2}\mathbf{g}_a^{\times}\Delta t^2 & \mathbf{1}\Delta t & \mathbf{1} & \mathbf{\Phi}_{k,1}^{34} & \mathbf{\Phi}_{k,1}^{35} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \tag{125}$$

with the following form of the blocks

$$\mathbf{\Phi}_{k,1}^{14} = \int_{t_1}^{t_k}\mathbf{C}(\tau)\mathrm{d}\tau \tag{126}$$

$$\mathbf{\Phi}_{k,1}^{34} = -\int_{t_1}^{t_k} \left(\mathbf{r}\left(\theta\right)^{\times}\mathbf{C}(\theta)\right) \tag{127}$$

$$+ \left[\int_{t_1}^{\theta}\left(\mathbf{v}(\tau)^{\times}\mathbf{C}(\tau) + \mathbf{g}_a^{\times}\int_{t_1}^{\tau}\mathbf{C}(s)\mathrm{d}s\right)\mathrm{d}\tau\right]\mathrm{d}\theta, \tag{128}$$

$$\mathbf{\Phi}_{k,1}^{35} = -\int_{t_1}^{t_k}\left(\int_{t_1}^{\tau}\left(\mathbf{C}(s)\mathrm{d}s\right)\right)\mathrm{d}\tau, \tag{129}$$

$$\mathbf{\Phi}_{k,1}^{24} = -\int_{t_1}^{t_k}\left(\mathbf{v}(\tau)^{\times}\mathbf{C}(\tau) + \mathbf{g}_a^{\times}\int_{t_1}^{\tau}\mathbf{C}(s)\mathrm{d}s\right)\mathrm{d}\tau, \tag{130}$$

$$\mathbf{\Phi}_{k,1}^{25} = \mathbf{\Phi}_{14} = -\int_{t_1}^{t_k}\mathbf{C}(\tau)\mathrm{d}\tau. \tag{131}$$

This is consistent with the expressions for the discrete-time state transition matrix found in [5].