Lie Groups Summary

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1 Lie Group Basics

This document will serve as a summary of concepts from Lie groups and is based heavily off the Micro Lie Theory paper [1], with a few addition derivations and summaries.

A Lie group both encompasses the definitions of *group* and *manifold*. A Lie group G is a smooth manifold whose elements satisfy the group axioms. A differentiable or *smooth* manifold is a topological space that is locally Euclidean. In robotics, it is common to say that our state *evolves* on this surface, meaning that the manifold defines the constraints that are imposed on the state. The smoothness of the manifold implies the existence of a unique tangent space at each point, which is a linear or vector space on which we are allowed to do calculus.

Next, a *group* $(G, \circ)$ is a set G with a composition operation, $\circ$, that for elements $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in G$ all satisfy the following axioms,

$$\begin{aligned} \text{Closure Under } \circ: &\quad \mathcal{X} \circ \mathcal{Y} \in G, && (1)\\ \text{Identity } \mathcal{E}: &\quad \mathcal{E} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{E} = \mathcal{X}, && (2)\\ \text{Inverse}: &\quad \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{X}^{-1} \circ \mathcal{X} = \mathcal{E}, && (3)\\ \text{Associativity}: &\quad (\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z}). && (4) \end{aligned}$$

The group structure imposes that the composition of elements remains on the manifold, and that each element also has an inverse on the manifold. A special one of these elements is the identity, and thus, a special one of the tangent spaces is the tangent space at the identity element, which is called the Lie algebra of the Lie group.

1.1 Group Actions

Lie groups come with the power to transform other elements of other sets. Given a Lie group G and a set V, $\mathcal{X} \cdot v$ is denoted the *action* of $\mathcal{X} \in \mathcal{M}$ on $v \in V$, given by

$$\cdot : M \times V \to V, \quad (\mathcal{X}, v) \mapsto \mathcal{X} \cdot v. \tag{5}$$

For $\cdot$ to be a group action, it must satisfy the axioms

$$\begin{aligned} \text{Identity}: &\quad \mathcal{E} \cdot v = v, && (6)\\ \text{Compatibility}: &\quad (\mathcal{X} \circ \mathcal{Y}) \cdot v = \mathcal{X} \cdot (\mathcal{Y} \cdot v). && (7) \end{aligned}$$

1.2 Tangent Spaces and The Lie Algebra

Given $\mathcal{X}(t)$, a point moving on a Lie group's manifold G, it's velocity $\dot{\mathcal{X}} = \partial \mathcal{X} / \partial t$ belongs to the tangent space $T_{\mathcal{X}} G$. The smoothness of the manifold implies the existence of a unique tangent space at each point. The structure of such tangent spaces is the same everywhere.

The tangent space at the identity element is called the Lie algebra, and is denoted

$$\mathfrak{g} \triangleq T_{\mathcal{E}} G. \tag{8}$$

Every Lie group has an associated Lie algebra. The Lie group is related to the group through the following facts:

- The Lie algebra $\mathfrak{g}$ is a vector space, and it can be *identified* with vectors in $\mathbb{R}^m$, where m is the number of degrees of freedom of the group G.
- The *exponential map*, $\exp : \mathfrak{g} \to G$, exactly converts elements of the Lie algebra to elements of the Lie group. The *logarithmic map*, defined $\log : G \to \mathfrak{g}$, is the inverse operation.
- Vectors of the tangent space at $\mathcal{X}$ can be transformed to the tangent space at the identity $\mathcal{E}$ through a linear transformation. This transformaion is called the *adjoint*.

Lie algebras can be defined locally to a tangent point, $\mathcal{X}$, estabilishing local coordinates for $T_{\mathcal{X}}\mathcal{M}$. Elements of the Lie algebra are typically defined with a wedge, as $\mathbf{v}^\wedge$, for a velocity. The structure of the Lie algebra by time differentiating the group contraint for the inverse. For multiplicative groups, this yields the new constraint $\mathcal{X}^{-1}\dot{\mathcal{X}} + \dot{\mathcal{X}}^{-1}\mathcal{X} = 0$. This constraint applies to the elements tangent at $\mathcal{X}$.

1.3 The Cartesian Vector Space $\mathbb{R}^m$

The elements $\boldsymbol{\tau}$ often have non-trivial structures (skew-symmetric matrices, imaginary numbers, pure quaternions, etc.) However, the key aspect is that they can be expressed as linear combinations of some base elements E_i, where E_i are called the *generators* of $\mathfrak{m}$ These are derivatives of $\mathcal{X}$ aroudn the origin in the $i'th$ direction. We can pass from $\mathbb{R}^m$ and $\mathfrak{g}$ through two mutially inverse linear maps or *isomorphisms*. Recall that an isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an *inverse mapping*. These two isomorphisms are commonly denoted *hat* and *vee*, written as

$$(\cdot)^\vee : \mathfrak{g} \to \mathbb{R}^m, \tag{9}$$

$$(\cdot)^\wedge : \mathbb{R}^m \to \mathfrak{g}. \tag{10}$$

Since $\mathfrak{g}$ is isomorphic to $\mathbb{R}^m$, we can instead just use vectors in $\mathbb{R}^m$ for our purposes, since they can be stacked in larger state vectors and more importantly, they can be manipulated with linear algebra. Because of this, it is often preferable to work with elements of $\mathbb{R}^m$ over workin with $\mathfrak{g}$,

1.4 The Exponential Map

The exponential map $\exp()$ allows for the exact transfer elements of the Lie algebra to the Lie group, an operation known as *retraction*. The exponential map arises naturally, by considering the time-derivatives of $\mathcal{X} \in G$ over the manifold, as follows. The structure for the elements of the Lie algebra can be found by differentiating the group constraint, as

$$\mathbf{v}^\wedge = \mathcal{X}^{-1}\dot{\mathcal{X}} = -\dot{\mathcal{X}}^{-1}\mathcal{X}. \tag{11}$$

Isolating $\dot{\mathcal{X}}$ yields

$$\dot{\mathcal{X}} = \mathcal{X}\mathbf{v}^\wedge. \tag{12}$$

For $\mathbf{v}(t)$ constant over time, this is an ordinary differential equation (ODE) whose solution is given by

$$\mathcal{X}(t) = \mathcal{X}(0) \exp\left(\mathbf{v}^{\wedge} t\right). \tag{13}$$

Since $\mathcal{X}(t)$ and $\mathcal{X}(0)$ are elements of the group, then $\exp\left(\mathbf{v}^{\wedge} t\right)$ mapes elements $\mathbf{v}^{\wedge} t$ of the Lie algebra to elements of the Lie group.

In order to provide a more generic definition of the exponential map, first, define the tangent increment $\boldsymbol{\tau} \triangleq \mathbf{v}t \in \mathbb{R}^m$ as velocity per time, so we have that $\boldsymbol{\tau}^{\wedge} = \mathbf{v}^{\wedge} \in \mathfrak{g}$ is a point in the Lie algebra. The exponential and logarismic maps defined as $\exp : \mathfrak{g} \rightarrow \mathcal{G}$, and $\log : \mathcal{G} \rightarrow \mathfrak{g}$, such that

$$\mathcal{X} = \exp\left(\boldsymbol{\tau}^{\wedge}\right), \tag{14}$$

$$\boldsymbol{\tau}^{\wedge} = \log\left(\mathcal{X}\right). \tag{15}$$

Closed forms of the exponential map in multiplicative groups are obtained by writin the absolute convergent Taylor series,

$$\exp\left(\boldsymbol{\tau}^{\wedge}\right) = \mathcal{E} + \boldsymbol{\tau}^{\wedge} + \frac{1}{2}\boldsymbol{\tau}^{\wedge^2} + \frac{1}{6}\boldsymbol{\tau}^{\wedge^3} + \ldots. \tag{16}$$

To find closed form solutions, we take advantage of the properties of powers of $\boldsymbol{\tau}^{\wedge}$, and invert them to find expressions for the logarithmic map. Some key properties fo the exponential map are

$$\exp\left((t+s)\boldsymbol{\tau}^{\wedge}\right) = \exp\left(t\boldsymbol{\tau}^{\wedge}\right)\exp\left(s\boldsymbol{\tau}^{\wedge}\right), \tag{17}$$

$$\exp\left(t\boldsymbol{\tau}^{\wedge}\right) = \exp\left(\boldsymbol{\tau}\right)^{t}, \tag{18}$$

$$\exp\left(-\boldsymbol{\tau}^{\wedge}\right) = \exp\left(\boldsymbol{\tau}\right)^{-1}, \tag{19}$$

$$\exp\left(\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}\right) = \mathcal{X}\exp\left(\boldsymbol{\tau}^{\wedge}\right)\mathcal{X}^{-1}. \tag{20}$$

The last identity can be derived by expanding the Taylor series and simplifing the many terms $\mathcal{X}^{-1}\mathcal{X}$.

1.5 The $\mathrm{Exp}\left(\cdot\right)$, $\mathrm{Log}\left(\cdot\right)$, $\oplus$, and $\ominus$ operations

The capital $\mathrm{Exp}\left(\cdot\right)$ and $\mathrm{Log}\left(\cdot\right)$ maps are convenient shorcuts to map vector elements to group elements and vice versa. These are defined as $\mathrm{Exp} : \mathfrak{g} \rightarrow G$ and $\mathrm{Log} : G \rightarrow \mathfrak{g}$, such that

$$\mathcal{X} = \exp\left(\boldsymbol{\tau}^{\wedge}\right) \triangleq \mathrm{Exp}\left(\boldsymbol{\tau}\right), \tag{21}$$

$$\boldsymbol{\tau} = \log\left(\mathcal{X}\right) \triangleq \mathrm{Log}\left(\mathcal{X}\right). \tag{22}$$

The general $\oplus$ and $\ominus$ operators allows us to introduce increments between elements of a curved manifold, and express them in its flat tangent vector space. Denoted by $\oplus$ and $\ominus$, they combine one Exp / Log operation with one composition. They have two possible definitions - left or right. These are given by

$$\mathcal{Y} = \mathcal{X} \oplus \boldsymbol{\tau} \triangleq \mathrm{Exp}\left(\boldsymbol{\tau}\right) \circ \mathcal{X}, \quad \text{(Lie group left)}, \tag{23}$$

$$\mathcal{Y} = \mathcal{X} \oplus \boldsymbol{\tau} \triangleq \mathcal{X} \circ \mathrm{Exp}\left(\boldsymbol{\tau}\right), \quad \text{(Lie group right)}, \tag{24}$$

$$\tag{25}$$

For subtraction, the left and right-minus operations are corresponding defined as

$$\mathcal{Y} \ominus \mathcal{X} \triangleq \operatorname{Log}\left(\mathcal{Y} \circ \mathcal{X}^{-1}\right), \quad \text{(Lie group left)}, \tag{26}$$
$$\mathcal{Y} \ominus \mathcal{X} \triangleq \operatorname{Log}\left(\mathcal{X}^{-1} \circ \mathcal{Y}\right), \quad \text{(Lie group right)}. \tag{27}$$

Note that these are simply obtained by rearranging the definitions of $\oplus$ for both the right and the left cases.

In the right definition of the $\oplus$ operator, $\operatorname{Exp}(\boldsymbol{\tau})$ is applied to the right hand side of the composition, meaning tat $\boldsymbol{\tau}$ belongs to the tangent space at $\mathcal{X}$ - it is often said that $\boldsymbol{\tau}$ is expressed in the *local* frame. In the left definition, $\operatorname{Exp}(\boldsymbol{\tau})$ occurs on the left and this is a perturbation at the tangent space at the identity element. We say that this perturbation is expressed in the global frame.

For elements of *matrix* Lie groups, the $\circ$ operator is simply matrix multiplication, and hence, the left and right definitions of the $\oplus$ operator are given by

$$\mathbf{Y} = \operatorname{Exp}(\boldsymbol{\tau})\,\mathbf{X}, \tag{28}$$
$$\mathbf{Y} = \mathbf{X}\operatorname{Exp}(\boldsymbol{\tau}). \tag{29}$$

These can be used, for example, to define uncertainty representations on Lie groups, either as

$$\mathbf{X} = \operatorname{Exp}(\delta\boldsymbol{\xi})\bar{\mathbf{X}}, \quad \text{Matrix Lie group left}, \tag{30}$$
$$\mathbf{X} = \bar{\mathbf{X}}\operatorname{Exp}(\delta\boldsymbol{\xi}), \quad \text{Matrix Lie group right}. \tag{31}$$

Note that this leads to the following error definitons corresponding to the left and right perturbations:

$$\delta\boldsymbol{\xi} = \mathbf{X} \ominus \bar{\mathbf{X}} = \operatorname{Log}\left(\mathbf{X}\bar{\mathbf{X}}^{-1}\right), \quad \text{Matrix Lie group left}, \tag{32}$$
$$\delta\boldsymbol{\xi} = \mathbf{X} \ominus \bar{\mathbf{X}} = \operatorname{Log}\left(\bar{\mathbf{X}}^{-1}\mathbf{X}\right), \quad \text{Matrix Lie group right}. \tag{33}$$

1.6 The Adjoint and the Adjoint Matrix

Equating $\mathcal{Y}$ in the definition of both the left and right definitions of the $\oplus$ operations, we arrive at $\operatorname{Exp}\left(\boldsymbol{\tau}^{\mathcal{E}}\right) \circ \mathcal{X} = \mathcal{X}\operatorname{Exp}\left(\boldsymbol{\tau}^{\mathcal{X}}\right)$, which determines a relation between the local and global tangent elements. This is then developed as

$$\operatorname{Exp}\left(\boldsymbol{\tau}^{\mathcal{E}}\right) = \mathcal{X}\operatorname{Exp}\left(\boldsymbol{\tau}^{\mathcal{X}}\right)\mathcal{X}^{-1} = \exp\left(\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}\right), \boldsymbol{\tau}^{\mathcal{E}} = \mathcal{X}\boldsymbol{\tau}^{\mathcal{X}\wedge}\mathcal{X}^{-1}. \tag{34}$$

The *adjoint* of G at $\mathcal{X}$, denoted $\operatorname{Ad}_{\mathcal{X}} : \mathfrak{g} \to \mathfrak{g}$, is defined as

$$\operatorname{Ad}_{\mathcal{X}}\left(\boldsymbol{\tau}^{\wedge}\right) \triangleq \mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}. \tag{35}$$

This means that $\boldsymbol{\tau}^{\mathcal{E}} = \operatorname{Ad}_{\mathcal{X}}\left(\boldsymbol{\tau}^{\mathcal{X}}\right)$. This defines the *adjoint action* of the group on its own Lie algebra. The adjoint has two interesting and easy to prove properties - it is linear, and homomorphism. Since $\operatorname{Ad}_{\mathcal{X}}$ is a linear map, we can find an equivalent matrix representation $\operatorname{Ad}(\mathcal{X})$ that maps the Cartesian tangent vectors as

$$\boldsymbol{\tau}^{\mathcal{E}} = \operatorname{Ad}(\mathcal{X})\,\boldsymbol{\tau}^{\mathcal{X}}, \tag{36}$$

where $\mathrm{Ad}\left(\mathcal{X}\right) \in \mathbb{R}^{m\times m}$ is the adjoint matrix. By the definition of the adjoint,

$$\mathcal{X}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1} = \left(\mathrm{Ad}\left(\mathcal{X}\right)\boldsymbol{\tau}\right)^\wedge, \tag{37}$$
$$\exp\left(\mathcal{X}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1}\right) = \exp\left(\left(\mathrm{Ad}\left(\mathcal{X}\right)\boldsymbol{\tau}\right)^\wedge\right), \tag{38}$$
$$\mathcal{X}\exp\left(\boldsymbol{\tau}^\wedge\right)\mathcal{X}^{-1} = \exp\left(\left(\mathrm{Ad}\left(\mathcal{X}\right)\boldsymbol{\tau}\right)^\wedge\right), \tag{39}$$

which is a key identity involving the adjoint matrix.

1.7 Derivatives on Lie Groups

The Jacobians described here fulfill the chain rule, so that we can easily compute any Jacobian from the partial Jacobian blocks of *inversion*, *composition*, *exponentiation*, and *action*. Beginning with Jaobians on vector spaces, recall that for a multivariate function $f : \mathbb{R}^m \to \mathbb{R}^n$, the Jacobian matrix is defined as the $n \times m$ matrix stacking all partial derivatives:

$$\mathbf{J} = \frac{\partial f\left(\mathbf{x}\right)}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} \in \mathbb{R}^{n\times m}. \tag{40}$$

It is useful to define this matrix as $\mathbf{J} = \left[\mathbf{j}_1 \ldots \mathbf{j}\right]$, where $\mathbf{j}_i$ is the $i'th$ column vector of the Jacobian. This column matrix corresponds to

$$\mathbf{j}_i = \frac{\partial f\left(\mathbf{x}\right)}{\partial x_i} \triangleq \lim_{h\to 0}\frac{f\left(\mathbf{x}+h\mathbf{e}_i\right) - f\left(\mathbf{x}\right)}{h}, \tag{41}$$

The numerator corresponds to the variation of $f\left(\mathbf{x}\right)$when $\mathbf{x}$ is perturbed in the direction of $\mathbf{e}_i$. For the sake of convenience, consider the compact form given by

$$\mathbf{J} = \frac{\partial f\left(\mathbf{x}\right)}{\partial \mathbf{x}} = \lim_{h\to 0}\frac{f\left(\mathbf{x}+\mathbf{h}\mathbf{e}\right) - f\left(\mathbf{x}\right)}{\mathbf{h}}, \tag{42}$$

with $\mathbf{h} \in \mathbb{R}^m$, which aglutinates all columns to form the definition of the Jacobian.

1.7.0.1 Right Jacobians on Lie Groups Inspired by the standard definitions of the $\oplus$ and $\ominus$ operators, Jacobians of functions $f : M \to N$ which act on manifolds can be derived. Using the right definitions of the $\oplus$ and $\ominus$ operators in place of $+$ and $-$, the standard derivative is formed as

$$\frac{Df\left(\mathcal{X}\right)}{D\mathcal{X}} \triangleq \lim_{\boldsymbol{\tau}\to 0}\frac{f\left(\mathcal{X}\oplus\boldsymbol{\tau}\right)\ominus f\left(\mathcal{X}\right)}{\boldsymbol{\tau}}, \tag{43}$$
$$= \frac{\partial\,\mathrm{Log}\left(f\left(\mathcal{X}\right)^{-1}\circ f\left(\mathcal{X}\circ\mathrm{Exp}\left(\boldsymbol{\tau}\right)\right)\right)}{\partial}\boldsymbol{\tau}. \tag{44}$$

Note that this is actually just the standard derivative of the rather complex function $g\left(\boldsymbol{\tau}\right) = \mathrm{Log}\left(f\left(\mathcal{X}\right)^{-1}\circ f\left(\mathrm{Exp}\left(\boldsymbol{\tau}\right)\circ\mathcal{X}\right)\right)$. With the $\oplus$ and $\ominus$, the intuition becomes much clearer - this is the Jacobian of $f\left(\mathcal{X}\right)$ with respect to $\mathcal{X}$, only we've expressed the infintesimal variations in the tangent spaces. Now, variations in $\mathcal{X}$ and $f(\mathcal{X})$ are expressed as vectors in their local tangent spaces, i.e., tangent respectively at $\mathcal{X} \in M$ and $f(\mathcal{X}) \in N$. This derivative is a JAcobian matrix $\in \mathbb{R}^{n\times m}$ mapping the local tangent spaces. For small values of $\boldsymbol{\tau}$, the following first order approximation holds.

$$f(\mathcal{X} \oplus \boldsymbol{\tau}) \approx f(\mathcal{X}) \oplus \frac{Df(\mathcal{X})}{D\mathcal{X}}\boldsymbol{\tau}. \tag{45}$$

1.7.0.2 Left Jacobians on Lie Groups Derivates can also be defined utilizing the left definitions of the $\oplus$ and $\ominus$ operators, which yields

$$\frac{Df(\mathcal{X})}{D\mathcal{X}} \triangleq \lim_{\boldsymbol{\tau}\to 0} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}}, \tag{46}$$

$$= \frac{\partial \operatorname{Log}\left(f(\operatorname{Exp}(\boldsymbol{\tau}) \circ \mathcal{X}) \circ f(\mathcal{X})^{-1}\right)}{\partial \boldsymbol{\tau}}. \tag{47}$$

Now, note that the left Jacobian is a matrix mapping variations in the *global* tangent spaces. For small values of $\boldsymbol{\tau}$, (45) still holds, but now the left definition of $\oplus$ must be used instead of the right definition of $\oplus$.

1.8 Uncertainty on Manifolds and Covariance Propagation

Local perturbations about a point $\bar{\mathcal{X}} \in M$ can be defined using using the right $\oplus$ and $\ominus$ operator as

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \boldsymbol{\tau}, \qquad \boldsymbol{\tau} = \mathcal{X} \ominus \bar{\mathcal{X}} \in T_{\bar{\mathcal{X}}}M. \tag{48}$$

Covariances can be properly defined on this tangent space through the standard expectation operator $\mathbb{E}[\cdot]$, written as

$$\boldsymbol{\Sigma}_{\mathcal{X}} \triangleq \mathbb{E}[\boldsymbol{\tau}\boldsymbol{\tau}^{\mathsf{T}}] = \mathbb{E}[\left(\boldsymbol{X} \ominus \bar{\boldsymbol{X}}\right)\left(\boldsymbol{X} \ominus \bar{\boldsymbol{X}}\right)^{\mathsf{T}}] \in \mathbb{R}^{m\times m}., \tag{49}$$

This allows us to define Gaussian variables on manifolds. Note that the covariance is actually that of the tangent perturbation $\boldsymbol{\tau}$ By using a left definition of the $\oplus$ and $\ominus$ operator, we can also define perturbations in the global reference (meaning the tangent space at the origin), as

$$\mathcal{X} = \operatorname{Exp}(\boldsymbol{\tau})\boldsymbol{\tau}, \qquad \boldsymbol{\tau} = \mathcal{X} \ominus \bar{\mathcal{X}}. \tag{50}$$

This allows for global specification of covariance matrices using the definition of the left-minus. Since global and local perturbations are related by the adjoint, their covariance can be tranformed with

$$\boldsymbol{\Sigma}_{\mathcal{X}}^{\mathcal{E}} = \mathbf{Ad}_{\mathcal{X}}\boldsymbol{\Sigma}_{\mathcal{X}}^{\mathcal{X}}\mathbf{Ad}_{\mathcal{X}}^{\mathsf{T}}. \tag{51}$$

Covariance propagation through a function $\mathcal{Y} = f(\mathcal{X})$ requires the linearization to yield the following formula

$$\boldsymbol{\Sigma}_{\mathcal{Y}} \approx \frac{Df}{D\mathcal{X}}\boldsymbol{\Sigma}_{\mathcal{X}}\frac{Df}{D\mathcal{X}}^{\mathsf{T}}. \tag{52}$$

1.9 Differentiation Rules on Manifolds

For all the typical manifolds G that we use, closed form solutions for the elementary Jacobians of *inversion*, *composition*, *exponentiation*, and *action* can be defined. The Jacobians will first be developed using right-Jacobians, using the Jacobian definition given by (43). Also, recall that the left and right Jacobians are related through the adjoint, written as

$$\frac{{}^{\mathcal{E}}Df\left(\mathcal{X}\right)}{D\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})}\frac{{}^{\mathcal{X}}Df\left(\mathcal{X}\right)}{D\mathcal{X}}\mathbf{Ad}_{\mathcal{X}}^{-1}. \tag{53}$$

Note that the notation used here will also reflect

$$\mathbf{J}_{\mathcal{X}}^{f(\mathcal{X})} \triangleq= \frac{Df\left(\mathcal{X}\right)}{D\mathcal{X}}, \qquad \mathbf{J}_{\mathcal{X}}^{\mathcal{Y}} \triangleq \frac{D\mathcal{Y}}{D\mathcal{X}}. \tag{54}$$

1.9.1 The Chain Rule

Letting $\mathcal{Y} = f(\mathcal{X})$ and $\mathcal{Z} = g(\mathcal{Y})$, we have that $\mathcal{Z} = g\left(f\left(\mathcal{X}\right)\right)$. The chain rule states

$$\frac{D\mathcal{Z}}{D\mathcal{X}} = \frac{D\mathcal{Z}}{D\mathcal{Y}}\frac{D\mathcal{Y}}{D\mathcal{X}} = \mathbf{J}_{\mathcal{Y}}^{\mathcal{Z}}\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}}. \tag{55}$$

1.9.2 Inverse

Start by defining

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} \triangleq \frac{D\mathcal{X}^{-1}}{D\mathcal{X}}, \tag{56}$$

Taking the derivative of this function using the right definition of the Jacobian yields

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} = \lim_{\boldsymbol{\tau}\to 0} = \frac{\mathrm{Log}\left(\mathcal{X}\left(\mathcal{X}\,\mathrm{Exp}\left(\boldsymbol{\tau}\right)\right)^{-1}\right)}{\boldsymbol{\tau}} \tag{57}$$

$$= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}\left(\mathcal{X}\,\mathrm{Exp}\left(-\boldsymbol{\tau}\right)\mathcal{X}^{-1}\right)}{\boldsymbol{\tau}}. \tag{58}$$

Next, utiliing the property $\exp\left(\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}\right) = \mathcal{X}\exp\left(\boldsymbol{\tau}^{\wedge}\right)\mathcal{X}^{-1}$, this is written as

$$\lim_{\boldsymbol{\tau}\to 0} = \frac{\left(-\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}\right)^{\wedge}}{\boldsymbol{\tau}} = -\mathbf{Ad}_{\mathcal{X}}. \tag{59}$$

Next, taking the derivative of this function using the left definition of the Jacobian yields

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} = \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}\left(\mathcal{X}^{-1}\,\mathrm{Exp}\left(-\boldsymbol{\tau}\right)\mathcal{X}\right)}{\boldsymbol{\tau}}, \tag{60}$$

$$= -\mathbf{Ad}_{\mathcal{X}^{-1}}. \tag{61}$$

This can also be confirmed by the relationship between derivatives as stated in (53) - plugging in the right Jacobian yields

$$\frac{{}^{\mathcal{E}}Df(\mathcal{X})}{D\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})}. \tag{62}$$

1.9.3 Composition

Define the following Jacobians

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\circ\mathcal{Y}} \triangleq \frac{{}^{\mathcal{X}}D\mathcal{X}\circ\mathcal{Y}}{D\mathcal{X}}, \qquad \mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} \triangleq \frac{{}^{\mathcal{Y}}D\mathcal{X}\circ\mathcal{Y}}{D\mathcal{Y}}. \tag{63}$$

Starting with a right definition of the Lie Jacobian, we arrive at

$$\begin{aligned}
\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\circ\mathcal{Y}} &= \lim_{\boldsymbol{\tau}\to 0} \frac{\operatorname{Log}\left((\mathcal{X}\mathcal{Y})^{-1}\mathcal{X}\operatorname{Exp}(\boldsymbol{\tau})\mathcal{Y}\right)}{\boldsymbol{\tau}}, & (64)\\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\operatorname{Log}\left(\mathcal{Y}^{-1}\operatorname{Exp}(-\boldsymbol{\tau})\mathcal{Y}\right)}{\boldsymbol{\tau}}, & (65)\\
&= \mathbf{Ad}_{\mathcal{Y}^{-1}}. & (66)
\end{aligned}$$

And for the Jacobian $\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}}$,

$$\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}} = \mathbf{1}. \tag{67}$$

Utilizing a left definition of the Jacobian yields

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\circ\mathcal{Y}} = \lim_{\boldsymbol{\tau}\to 0} = \frac{\operatorname{Log}\left(\operatorname{Exp}(\boldsymbol{\tau})\mathcal{X}\mathcal{Y}\mathcal{Y}^{-1}\mathcal{X}^{-1}\right)}{\boldsymbol{\tau}} = \mathbf{1}. \tag{68}$$

and for $\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}}$,

$$\begin{aligned}
\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}} &= \frac{\partial \operatorname{Log}\left((\mathcal{X}\operatorname{Exp}(\boldsymbol{\tau})\mathcal{Y})(\mathcal{X}\mathcal{Y})^{-1}\right)}{\partial\boldsymbol{\tau}} & (69)\\
&= \frac{\partial \operatorname{Log}\left(\mathcal{X}\operatorname{Exp}(\boldsymbol{\tau})\mathcal{X}^{-1}\right)}{\partial\boldsymbol{\tau}}. & (70)\\
&= \mathbf{Ad}_{\mathcal{X}}. & (71)
\end{aligned}$$

1.9.4 Group Jacobians

The *right Jacobian of G* is defined as the right Jacobian of the function

$$\mathbf{J}_r(\boldsymbol{\tau}) = \frac{D\operatorname{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}. \tag{72}$$

The right Jacobian maps variations of the argument $\boldsymbol{\tau}$ into variations in the *local* tangent space at $\operatorname{Exp}(\boldsymbol{\tau})$. The actual definition of this Jacobian is given by

$$\begin{aligned}
\mathbf{J}_r(\boldsymbol{\tau}) &\triangleq \lim_{\boldsymbol{\tau}\to\mathbf{0}} \frac{\operatorname{Log}\left(\operatorname{Exp}(\boldsymbol{\tau})^{-1}\operatorname{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau})\right)}{\delta\boldsymbol{\tau}} & (73)\\
\mathbf{J}_r(\boldsymbol{\tau}) &= \frac{\partial \operatorname{Log}\left(\operatorname{Exp}(\boldsymbol{\tau})^{-1}\operatorname{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau})\right)}{\partial\delta\boldsymbol{\tau}}. & (74)
\end{aligned}$$

Note that this is actually just a regular vectorspace Jacobian and can be computed numerically! Inutitively, the right Jacobian measurements how the difference between $\mathrm{Exp}\left(\boldsymbol{\tau}\right)$ and$\mathrm{Exp}\left(\boldsymbol{\tau}+\delta\boldsymbol{\tau}\right)$, mapped back to $\mathbb{R}^m$, change with $\delta\boldsymbol{\tau}$.

For small variations of $\boldsymbol{\tau}$, we can use the first order approximation of $f\left(\boldsymbol{\tau}\right)=\mathrm{Exp}\left(\boldsymbol{\tau}\right)$ to write

$$\mathrm{Exp}\left(\boldsymbol{\tau}+\delta\boldsymbol{\tau}\right)\approx\mathrm{Exp}\left(\boldsymbol{\tau}\right)\mathrm{Exp}\left(\mathbf{J}_r\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}\right). \tag{75}$$

Defining a new quantity $\delta\boldsymbol{\tau}'=\mathbf{J}_r\left(\boldsymbol{\tau}\right)^{-1}\delta\boldsymbol{\tau}$, and performing the same operations yields

$$\mathrm{Exp}\left(\boldsymbol{\tau}+\delta\boldsymbol{\tau}'\right)\approx\mathrm{Exp}\left(\boldsymbol{\tau}\right)\mathrm{Exp}\left(\mathbf{J}_r\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}'\right), \tag{76}$$

$$\mathrm{Exp}\left(\boldsymbol{\tau}+\mathbf{J}_r\left(\boldsymbol{\tau}\right)^{-1}\delta\boldsymbol{\tau}\right)=\mathrm{Exp}\left(\boldsymbol{\tau}\right)\mathrm{Exp}\left(\delta\boldsymbol{\tau}\right), \tag{77}$$

$$\mathrm{Log}\left(\mathrm{Exp}\left(\boldsymbol{\tau}\right)\mathrm{Exp}\left(\delta\boldsymbol{\tau}\right)\right)\approx\boldsymbol{\tau}+\mathbf{J}_r\left(\boldsymbol{\tau}\right)^{-1}\delta\boldsymbol{\tau}. \tag{78}$$

Similarly, the *left Jacobian of G* is defined as the left Jacobian of the function $\mathcal{X}=\mathrm{Exp}\left(\boldsymbol{\tau}\right)$ and is defined as

$$\mathbf{J}_l\left(\boldsymbol{\tau}\right)=\frac{D\,\mathrm{Exp}\left(\boldsymbol{\tau}\right)}{D\boldsymbol{\tau}}. \tag{79}$$

By plugging in this function $\mathcal{X}=\mathrm{Exp}(\boldsymbol{\tau})$ into the generic definition for the left Jacobian, the definition of the left Jacobian is given by

$$\mathbf{J}_l\left(\boldsymbol{\tau}\right)=\frac{\partial\,\mathrm{Log}\left(\mathrm{Exp}\left(\boldsymbol{\tau}+\delta\boldsymbol{\tau}\right)Exp\left(\boldsymbol{\tau}\right)^{-1}\right)}{\partial\delta\boldsymbol{\tau}}. \tag{80}$$

This leads to similar approximations for small $\delta\boldsymbol{\tau}$ that involve the left Jacobian, as

$$\mathrm{Exp}\left(\boldsymbol{\tau}+\delta\boldsymbol{\tau}\right)\approx\mathrm{Exp}\left(\mathbf{J}_l\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}\right)\mathrm{Exp}\left(\boldsymbol{\tau}\right). \tag{81}$$

Equationg the two derived expressions for $\mathrm{Exp}\left(\boldsymbol{\tau}+\delta\boldsymbol{\tau}\right)$ yields

$$\mathrm{Exp}\left(\mathbf{J}_l\left(\tau\right)\delta\boldsymbol{\tau}\right)\mathrm{Exp}\left(\boldsymbol{\tau}\right)=\mathrm{Exp}\left(\boldsymbol{\tau}\right)\mathrm{Exp}\left(\mathbf{J}_r\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}'\right) \tag{82}$$

$$\mathrm{Exp}\left(\mathbf{J}_l\left(\tau\right)\delta\boldsymbol{\tau}\right)=\mathcal{X}\,\mathrm{Exp}\left(\mathbf{J}_r\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}\right)\mathcal{X}^{-1} \tag{83}$$

$$=\mathrm{Exp}\left(\mathcal{X}\mathbf{J}_r\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}\mathcal{X}^{-1}\right), \tag{84}$$

$$\mathbf{J}_l\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}=\mathcal{X}\mathbf{J}_r\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}\mathcal{X}^{-1}, \tag{85}$$

and hence, the left and right Jacobians are related through the adjoint as

$$\mathbf{J}_l\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}=\mathbf{Ad}_{\mathcal{X}}\mathbf{J}_r\left(\boldsymbol{\tau}\right)\delta\boldsymbol{\tau}, \tag{86}$$

$$\mathbf{Ad}_{\mathcal{X}}=\mathbf{J}_l\left(\boldsymbol{\tau}\right)\mathbf{J}_r^{-1}\left(\boldsymbol{\tau}\right). \tag{87}$$

The chain rule also allows to to relate $\mathbf{J}_r$ and $\mathbf{J}_l$, and the relationship between these is

$$\mathbf{J}_r\left(\boldsymbol{\tau}\right)=\mathbf{J}_l\left(-\boldsymbol{\tau}\right). \tag{88}$$

1.9.5 Logarithmic Map

For $\boldsymbol{\tau} = \text{Log}(\mathcal{X})$, and for the definition of the right Jacobian, from (78),

$$\mathbf{J}_{\mathcal{X}}^{\text{Log}(\mathcal{X})} = \mathbf{J}_r^{-1}(\boldsymbol{\tau}). \tag{89}$$

For the same function $\boldsymbol{\tau} = \text{Log}(\mathcal{X})$ but this time using the definition of the left Jacobian, we get

$$\mathbf{J}_{\mathcal{X}}^{\text{Log}(\mathcal{X})} = \mathbf{J}_l^{-1}(\boldsymbol{\tau}). \tag{90}$$

1.9.6 $\oplus$ Operator

For the $\oplus$ operations, there are two Jacobians to be solved for: $\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\oplus\boldsymbol{\tau}}$ and $\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}}$.

Starting with the right definition of $\oplus$, and defining $f(\mathcal{X}) = \mathcal{X}\,\text{Exp}(\boldsymbol{\tau})$,

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\oplus\boldsymbol{\tau}} = \mathbf{J}_{\mathcal{X}}^{\mathcal{X}\circ\text{Exp}(\boldsymbol{\tau})} = \mathbf{Ad}_{\text{Exp}(\boldsymbol{\tau})}^{-1}, \tag{91}$$

using the result of the Jacobian of the composition operation. To derive $\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}}$, defining $\mathcal{Y} = \text{Exp}(\boldsymbol{\tau})$ we can use the chain rule to write

$$\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}} = \mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}}\mathbf{J}_{\boldsymbol{\tau}}^{\text{Exp}(\boldsymbol{\tau})} = \mathbf{1}\mathbf{J}_r = \mathbf{J}_r(\boldsymbol{\tau}). \tag{92}$$

The same can be repeated for the left definition of $\oplus$, and defining $f(\mathcal{X}) = \text{Exp}(\boldsymbol{\tau})\,\mathcal{X}$, these Jacobians are derived as

$$\mathbf{J}_{\mathcal{X}}^{\text{Exp}(\boldsymbol{\tau})\circ\mathcal{X}} = \mathbf{1}, \tag{93}$$

again using the results of the composition Jacobian. Next, for $\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}}$, we have

$$\mathbf{J}_{\boldsymbol{\tau}}^{\text{Exp}(\boldsymbol{\tau})\circ\mathcal{X}} = \mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\mathcal{X}}\mathbf{J}_{\boldsymbol{\tau}}^{\text{Exp}(\boldsymbol{\tau})} = \mathbf{J}_l, \tag{94}$$

using the results from the composition Jacobian and the group Jacobian for the left definition of $\oplus$.

1.9.7 $\ominus$ Operator

For the definition of the $\ominus$ operator, consider the functions

$$\mathcal{Z} = \mathcal{X}^{-1}\mathcal{Y}, \quad \boldsymbol{\tau} = \mathcal{Y}\ominus\mathcal{X} = \text{Log}(\mathcal{Z}). \tag{95}$$

Consider the output of the minus operation, given by

$$\boldsymbol{\tau} = \mathcal{Y}\ominus\mathcal{X}. \tag{96}$$

Starting with $\mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\ominus\mathcal{X}}$, and utilizing a right-perturbation yields

$$\begin{aligned}
\boldsymbol{\tau} + \delta\boldsymbol{\tau} &= (\mathcal{Y}\oplus\delta\mathbf{y})\ominus\mathcal{X}, && (97)\\
&= \text{Log}\left(\mathcal{X}^{-1}\mathcal{Y}\,\text{Exp}(\delta\mathbf{y})\right) && (98)\\
&= \text{Log}\left(\text{Exp}(\boldsymbol{\tau})\,\text{Exp}(\delta\mathbf{y})\right) && (99)\\
&\approx \boldsymbol{\tau} + \mathbf{J}_r^{-1}(\boldsymbol{\tau})\,\delta\mathbf{y}, && (100)
\end{aligned}$$

and hence, $\mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\ominus\mathcal{X}} = \mathbf{J}_r^{-1}(\boldsymbol{\tau})$ for a right perturbation. For a left perturbation, similarly,

$$\boldsymbol{\tau} + \delta\boldsymbol{\tau} = \operatorname{Log}\left(\operatorname{Exp}(\delta\mathbf{y})\,\mathcal{Y}\mathcal{X}^{-1}\right), \tag{101}$$
$$= \operatorname{Log}\left(\operatorname{Exp}(\delta\mathbf{y})\operatorname{Exp}(\boldsymbol{\tau})\right) \tag{102}$$
$$=\approx \boldsymbol{\tau} + \mathbf{J}_l^{-1}(\boldsymbol{\tau})\,\delta\mathbf{y}. \tag{103}$$

Next, to derive the Jacobian of $\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}}$ utilizing a right perturbation,

$$\boldsymbol{\tau} + \delta\boldsymbol{\tau} = \mathcal{Y} \ominus (\mathcal{X} \oplus \delta\mathbf{x}), \tag{104}$$
$$= \operatorname{Log}\left((\mathcal{X}\operatorname{Exp}(\delta\mathbf{x}))^{-1}\mathcal{Y}\right), \tag{105}$$
$$= \operatorname{Log}\left(\operatorname{Exp}(-\delta\mathbf{x})\,\mathcal{X}^{-1}\mathcal{Y}\right), \tag{106}$$
$$= \operatorname{Log}\left(\operatorname{Exp}(-\delta\mathbf{x})\operatorname{Exp}(\boldsymbol{\tau})\right), \tag{107}$$
$$\approx \boldsymbol{\tau} - \mathbf{J}_l(\boldsymbol{\tau})^{-1}\,\delta\mathbf{x}, \tag{108}$$

and hence, $\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}} = -\mathbf{J}_l(\boldsymbol{\tau})^{-1}$ utilizing the right perturbation.

Finally, deriving $\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}}$ using a left perturbation yields

$$\boldsymbol{\tau} + \delta\boldsymbol{\tau} = \mathcal{Y} \ominus (\mathcal{X} \oplus \delta\mathbf{x}), \tag{109}$$
$$= \operatorname{Log}\left(\mathcal{Y}\left(\operatorname{Exp}(\delta\mathbf{x})\,\mathcal{X}\right)^{-1}\right), \tag{110}$$
$$= \operatorname{Log}\left(\mathcal{Y}\mathcal{X}^{-1}\operatorname{Exp}(-\delta\mathbf{x})\right), \tag{111}$$
$$= \operatorname{Log}\left(\operatorname{Exp}(\boldsymbol{\tau})\operatorname{Exp}(-\delta\mathbf{x})\right) \tag{112}$$
$$\approx \boldsymbol{\tau} - \mathbf{J}_r(\boldsymbol{\tau})^{-1}\,\delta\mathbf{x}. \tag{113}$$

1.10 Elementary Jacobians Summary

The following is a summary of the elementary Jacobians derived in the previous section.

Operation	Right Perturbation Jacobians	Left Perturbation Jacobians
$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\oplus\boldsymbol{\tau}}$	$\mathbf{Ad}_{\operatorname{Exp}(\boldsymbol{\tau})}^{-1}$	$\mathbf{1}$
$\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}}$	$\mathbf{J}_r(\boldsymbol{\tau})$	$\mathbf{J}_l(\boldsymbol{\tau})$
$\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}}$	$-\mathbf{J}_l(\boldsymbol{\tau})^{-1}$	$-\mathbf{J}_r(\boldsymbol{\tau})^{-1}$
$\mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\ominus\mathcal{X}}$	$\mathbf{J}_r^{-1}(\boldsymbol{\tau})$	$\mathbf{J}_l^{-1}(\boldsymbol{\tau})$

Table 1: Jacobian expressions for elementary Lie group operations under right- and left-invariant formulations.

1.11 Baker-Cambell-Hausdorff Formula

One of the main tools used to manipulate expressions involving Lie groups is the Baker-Cambell-Hausdorff (BCH) formula. Given $a, b \in \mathbb{R}$, the following holds.

$$\exp\left(a+b\right) = \exp(a)\exp(b) \tag{114}$$

Hwoever, with two matrices $\mathbf{A}, \mathbf{B} \in \mathfrak{g}$,

$$\exp\left(\mathbf{A}+\mathbf{B}\right) \neq \exp(\mathbf{A})\exp(\mathbf{B}). \tag{115}$$

The BCH formula relates $\exp\left(\mathbf{A}\right)$ and $\exp\left(\mathbf{B}\right)$, as

$$\mathbf{Z} \triangleq \log\left(\exp\left(\mathbf{A}\right)\exp\left(\mathbf{B}\right)\right) \in \mathfrak{g}. \tag{116}$$

The BCH formula can be expanded as

$$\mathbf{Z} = \log\left(\exp\left(\mathbf{A}\right)\exp\left(\mathbf{B}\right)\right) = \mathbf{A} + \mathbf{B} + \frac{1}{2}[\mathbf{A}, \mathbf{B}] + \frac{1}{12}[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] + \frac{1}{12}[\mathbf{B}, [\mathbf{A}, \mathbf{B}]] + \ldots, \tag{117}$$

where the *Lie bracket* is defined as

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}. \tag{118}$$

When $\mathbf{A}$ is “small”, the BCH formula becomes

$$\mathbf{Z} = \mathbf{b} + \mathbf{J}_\ell\left(\mathbf{b}\right)^{-1}\mathbf{a}. \tag{119}$$

When $\mathbf{B}$ is “small”, the BCH formula becomes

$$\mathbf{Z} = \mathbf{a} + \mathbf{J}_r\left(\mathbf{a}\right)^{-1}\mathbf{b}. \tag{120}$$

The previous results lead to approximations involving the left Jacobian, as

$$\exp\left(\left(\boldsymbol{\xi} + \delta\boldsymbol{\xi}\right)^\wedge\right) \approx \exp\left(\left(\mathbf{J}_\ell\left(\boldsymbol{\xi}\right)\delta\boldsymbol{\xi}\right)^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right), \tag{121}$$

$$\exp\left(\delta\boldsymbol{\xi}^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right) \approx \exp\left(\left(\boldsymbol{\xi} + \mathbf{J}_\ell\left(\boldsymbol{\xi}\right)^{-1}\delta\boldsymbol{\xi}\right)^\wedge\right), \tag{122}$$

$$\log\left(\exp\left(\delta\boldsymbol{\xi}^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right)\right)^\wedge \approx \boldsymbol{\xi} + \mathbf{J}_\ell\left(\boldsymbol{\xi}\right)^{-1}\delta\boldsymbol{\xi}, \tag{123}$$

where $\delta\boldsymbol{\xi}$ is small.

References

[1] J. Sola, J. Deray, and D. Atchuthan, “A Micro Lie Theory for State Estimation in Robotics,” *arXiv preprint arXiv:1812.01537*, 2018.