SLAM Sensor Models

Mitchell Cohen
Department of Mechanical Engineering, McGill University
817 Sherbrooke Street West, Montreal QC H3A 0C3

March 14, 2025

Contents

1 Preliminaries

This document contains derivations related to sensor models commonly used in robotic state estimation, especially in contexts related to SLAM. For each sensor model, Jacobians are derived with respect to the state variables, a task that is often critical for the use of the sensors in state estimation algorithms. Many of the sensor models involve estimating states that live on *Lie groups*, and two distinct options exist for deriving Jacobians on Lie groups. In this document, Jacobians are presented for both the "left" and "right" perturbations of the state, allowing for the use of these sensor models for both perturbation schemes.

Some preliminaries related to Lie groups, based largely on [1], are briefly covered before exploring common sensor models.

1.1 Lie Groups

A Lie group G is a smooth manifold that, given a group operation $\circ : G \times G \to G$, satisfy the group axioms. Given elements $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in G$, the group axioms are given by

$$\begin{aligned} \text{Closure Under } \circ : &\quad \mathcal{X} \circ \mathcal{Y} \in G, && (1) \\ \text{Identity } \mathcal{E} : &\quad \mathcal{E} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{E} = \mathcal{X}, && (2) \\ \text{Inverse} : &\quad \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{X}^{-1} \circ \mathcal{X} = \mathcal{E}, && (3) \\ \text{Associativity} : &\quad (\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z}). && (4) \end{aligned}$$

For any Lie group, there exists an associated Lie algebra $\mathfrak{g}$, a vector space identifiable with elements of $\mathbb{R}^m$, where m is referred to as the degrees of freedom of G. The Lie algebra is related to the group through the exponential and logarithmic maps, denoted $\exp : \mathfrak{g} \to G$ and $\log : G \to \mathfrak{g}$. The "vee" and "wedge" operators are denoted $(\cdot)^\vee : \mathfrak{g} \to \mathbb{R}^m$ and $(\cdot)^\wedge : \mathbb{R}^m \to \mathfrak{g}$, and are used to associate group elements with vectors with

$$\mathcal{X} = \exp(\boldsymbol{\xi}^\wedge) \triangleq \operatorname{Exp}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} = \log(\mathcal{X})^\vee \triangleq \operatorname{Log}(\mathcal{X}), \tag{5}$$

where $\mathcal{X} \in G, \boldsymbol{\xi} \in \mathbb{R}^m$. Note that throughout this document, the shorthand notation $\operatorname{Exp} : \mathbb{R}^m \to G$ and $\operatorname{Log} : G \to \mathbb{R}^m$ will be used.

This document will also make use of the general $\oplus$ and $\ominus$ operations, allowing for the introduction of increments to the curved manifold, expressed in its flat tangent vector space. The $\oplus$ and $\ominus$ operators combine one $\operatorname{Exp}(\cdot)$/ $\operatorname{Log}(\cdot)$ operation with one composition operation. They have two possible definitions, left or right. These are respectively given by

$$\begin{aligned} \mathcal{Y} = \mathcal{X} \oplus \boldsymbol{\tau} &\triangleq \operatorname{Exp}(\boldsymbol{\tau}) \circ \mathcal{X}, \quad \text{(Lie group left)}, && (6) \\ \mathcal{Y} = \mathcal{X} \oplus \boldsymbol{\tau} &\triangleq \mathcal{X} \circ \operatorname{Exp}(\boldsymbol{\tau}), \quad \text{(Lie group right)}, && (7) \end{aligned}$$

For subtraction, the left and right-minus operations are corresponding defined as

$$\begin{aligned} \mathcal{Y} \ominus \mathcal{X} &\triangleq \operatorname{Log}\left(\mathcal{Y} \circ \mathcal{X}^{-1}\right), \quad \text{(Lie group left)}, && (8) \\ \mathcal{Y} \ominus \mathcal{X} &\triangleq \operatorname{Log}\left(\mathcal{X}^{-1} \circ \mathcal{Y}\right), \quad \text{(Lie group right)}. && (9) \end{aligned}$$

Note that these are simply obtained by rearranging the definitions of $\oplus$ for both the right and the left cases.

For elements of *matrix* Lie groups, the $\circ$ operator is simply matrix multiplication, and hence, the left and right definitions of the $\oplus$ operator are given by

$$\begin{aligned} \mathbf{Y} &= \operatorname{Exp}(\boldsymbol{\tau}) \, \mathbf{X}, && (10) \\ \mathbf{Y} &= \mathbf{X} \operatorname{Exp}(\boldsymbol{\tau}), && (11) \end{aligned}$$

where $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n \times n}$ are matrices satisfying the group axioms, and $\boldsymbol{\tau} \in \mathbb{R}^m$ is isomorphic to the Lie algebra. These can be used, for example, to define uncertainty representations on Lie groups, either as

$$\mathbf{X} = \operatorname{Exp}(\delta\boldsymbol{\xi})\bar{\mathbf{X}}, \quad \text{Matrix Lie group left,} \tag{12}$$

$$\mathbf{X} = \bar{\mathbf{X}} \operatorname{Exp}(\delta\boldsymbol{\xi}), \quad \text{Matrix Lie group right,} \tag{13}$$

where $\delta\boldsymbol{\xi} \in \mathbb{R}^m$ is a small perturbation. Rearranging leads to the error definitions used for elements of a matrix Lie group, given by

$$\delta\boldsymbol{\xi} = \mathbf{X} \ominus \bar{\mathbf{X}} = \operatorname{Log}\left(\mathbf{X}\bar{\mathbf{X}}^{-1}\right), \quad \text{Matrix Lie group left,} \tag{14}$$

$$\delta\boldsymbol{\xi} = \mathbf{X} \ominus \bar{\mathbf{X}} = \operatorname{Log}\left(\bar{\mathbf{X}}^{-1}\mathbf{X}\right), \quad \text{Matrix Lie group right.} \tag{15}$$

1.2 Jacobians on Lie Groups

Following [1], the Jacobian of a function $f : M \to N$ with repsect to $\mathcal{X}$ can be derived. Using the definitions of $\oplus$ and $\ominus$ previously introduced, the Jacobian of f with respect to $\mathcal{X}$ evaluated at $\bar{\mathcal{X}}$ is written as

$$\left.\frac{Df(\mathcal{X})}{D\mathcal{X}}\right|_{\bar{\mathcal{X}}} \triangleq \left.\frac{\partial f(\bar{\mathcal{X}} \oplus \boldsymbol{\tau}) \ominus f(\bar{\mathcal{X}})}{\partial \boldsymbol{\tau}}\right|_{\boldsymbol{\tau}=\mathbf{0}}. \tag{16}$$

Utilizing this definition of a derivative on a Lie group leads to the definition of the *left and right group Jacobians*, which are defined as the Jacobians of the function $f(\boldsymbol{\tau}) = \operatorname{Exp}(\boldsymbol{\tau})$. This is written as

$$\mathbf{J}(\boldsymbol{\tau}) = \frac{D \operatorname{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}, \tag{17}$$

$$= \left.\frac{\partial \operatorname{Exp}(\bar{\boldsymbol{\tau}} + \boldsymbol{\tau}) \ominus \operatorname{Exp}(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}}\right|_{\boldsymbol{\tau}=\mathbf{0}}, \tag{18}$$

with the appropriate definition of the $\ominus$ operator. When a right definition of the $\ominus$ operator is used, the resultant Jacobian is the right group Jacobian and is denoted $\mathbf{J}_r(\boldsymbol{\tau})$. The left Jacobian is similarly denoted $\mathbf{J}_l(\boldsymbol{\tau})$, and utilizes a left definition of the $\ominus$ operator.

1.3 Common Lie Groups

This section briefly covers some common Lie groups found in robotic state estimation problems, as well as some useful definitions related to these groups.

1.3.1 The Special Orthogonal Group $SO(3)$

One of the most common Lie groups encountered in robotics is $SO(3)$, the set of three-dimentional rotations. This group is defined as

$$SO(3) = \left\{\mathbf{C} \in \mathbb{R}^{3\times 3} \mid \mathbf{C}\mathbf{C}^{\mathsf{T}} = \mathbf{1}, \det(\mathbf{C}) = +1\right\}. \tag{19}$$

The Lie algebra associated with $SO(3)$ is given by

$$\mathfrak{so}(3) = \left\{ \boldsymbol{\phi}^\times \in \mathbb{R}^{3\times3} | \boldsymbol{\phi} \in \mathbb{R}^3 \right\}, \tag{20}$$

where $\boldsymbol{\phi}^\times$ is a skew-symmetrix matrix given by

$$\boldsymbol{\phi}^\times = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}. \tag{21}$$

The exponential map from $\mathfrak{so}(3)$ to $SO(3)$ is written in closed form as

$$\operatorname{Exp}(\boldsymbol{\phi}) = \cos\phi \mathbf{1} + (1 - \cos\phi)\, \mathbf{a}\mathbf{a}^\mathsf{T} + \sin\phi \mathbf{a}^\times, \tag{22}$$

where $\mathbf{a} = \boldsymbol{\phi} / \|\boldsymbol{\phi}\|$ and $\phi = \|\boldsymbol{\phi}\|$. The left Jacobian of $SO(3)$ is given in closed form as

$$\mathbf{J}_l(\boldsymbol{\phi}) = \frac{\sin\phi}{\phi}\mathbf{1} + \frac{1-\cos\phi}{\phi}\mathbf{a}^\times + \left(1 - \frac{\sin\phi}{\phi}\right)\mathbf{a}\mathbf{a}^\mathsf{T}. \tag{23}$$

1.3.2 The Special Euclidean Group $SE(3)$

The *special Euclidean group* is often used to represent poses (i.e., position and orientation), and is defined as

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} \middle| \mathbf{C} \in SO(3), \mathbf{r} \in \mathbb{R}^3 \right\}. \tag{24}$$

The inverse of $\mathbf{T}$ is given by

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{C}^\mathsf{T} & -\mathbf{C}^\mathsf{T}\mathbf{r} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(3). \tag{25}$$

The Lie algebra associated with $SE(3)$ is given by

$$\mathfrak{se}(3) = \left\{ \boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge \in \mathbb{R}^{4\times4} \mid \boldsymbol{\xi} \in \mathbb{R}^6 \right\}, \tag{26}$$

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} \boldsymbol{\xi}^\phi \\ \boldsymbol{\xi}^\mathrm{r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi}^{\phi^\times} & \boldsymbol{\xi}^\mathrm{r} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}. \tag{27}$$

The exponential map from $\mathfrak{se}(3)$ to $SE(3)$ is written as

$$\operatorname{Exp}(\boldsymbol{\xi}) = \begin{bmatrix} \operatorname{Exp}(\boldsymbol{\xi}^\phi) & \mathbf{J}_l(\boldsymbol{\xi}^\phi)\,\boldsymbol{\xi}^\mathrm{r} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{28}$$

where $\mathbf{J}_l$ is the left Jacobian of $SO(3)$.

1.4 MAP Estimation and Nonlinear Least Squares

The *maximum a posteriori* (MAP) estimate is a point estimate that solves

$$\hat{\mathcal{X}}^{\mathrm{MAP}} = \arg\max_{\mathcal{X}} \frac{p\left(\mathcal{Y}|\mathcal{X}\right) p\left(\mathcal{X}\right)}{p\left(\mathcal{Y}\right)}, \tag{29}$$

where $p\left(\mathcal{Y}|\mathcal{X}\right)$ is known as the likelihood probability density function (PDF), $p\left(\mathcal{X}\right)$ is the prior PDF, and $p(\mathcal{Y})$ is the marginal PDF. Additionally, $\mathcal{X} = \left(\mathcal{X}_0, \ldots, \mathcal{X}_K\right)$ is a set of states to be estimated, and $\mathcal{Y} = \{\mathcal{Y}_0, \ldots \mathcal{Y}_n\}$ is a measurement set. Since $p\left(\mathcal{Y}\right)$ does not depend on $\mathcal{X}$, it is typically omitted to yield the equivalent optimization problem given by

$$\begin{aligned}\hat{\mathcal{X}} &= \arg\max_{\mathcal{X}} p\left(\mathcal{Y}|\mathcal{X}\right) p\left(\mathcal{X}\right) && (30)\\ &= \arg\max_{\mathcal{X}} p\left(\mathcal{X}_0\right) \prod_{i=1}^{n} p\left(\mathcal{Y}_i|\mathcal{X}_i\right). && (31)\end{aligned}$$

Here, the measurement likelihood, $p\left(\mathcal{Y}|\mathcal{X}\right)$, is factored into a product of individual likelihoods, $p\left(\mathcal{Y}_i|\mathcal{X}_i\right)$, for each measurement. Applying the negative log of this function which allows us to write the problem as a minimization, as

$$\hat{\mathcal{X}} = \arg\min_{\mathcal{X}} - \log p\left(\mathcal{X}_0\right) - \sum_{i=1}^{n} \log p\left(\mathcal{Y}_i|\mathcal{X}_i\right). \tag{32}$$

Consider the situation when $p\left(\mathcal{Y}_i|\mathcal{X}_i\right)$ is Gaussian, such that $p\left(\mathcal{Y}_i|\mathcal{X}_i\right) \sim \mathcal{N}\left(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i\right)$. The form of this Gaussian measurement likelihood is then given by

$$p\left(\mathcal{Y}_i|\mathcal{X}_i\right) = \frac{1}{\sqrt{(2\pi)^n \det\left(\Sigma_i\right)}} \exp\left(-\frac{1}{2}\left(\mathbf{e}_i\left(\mathcal{X}_i, \mathcal{Y}_i\right) - \boldsymbol{\mu}_i\right)^{\mathsf{T}} \boldsymbol{\Sigma}_i^{-1} \left(\mathbf{e}_i\left(\mathcal{X}_i, \mathcal{Y}_i\right) - \boldsymbol{\mu}_i\right)\right). \tag{33}$$

Dropping the constant term, this negative log-likelihood is written as

$$\hat{\mathcal{X}} = \arg\min_{\mathcal{X}} \frac{1}{2} \sum_{i=1}^{n} \left(\mathbf{e}_i\left(\mathcal{X}_i, \mathcal{Y}_i\right) - \boldsymbol{\mu}_i\right)^{\mathsf{T}} \boldsymbol{\Sigma}_i^{-1} \left(\mathbf{e}_i\left(\mathcal{X}_i, \mathcal{Y}_i\right) - \boldsymbol{\mu}_i\right). \tag{34}$$

Typically, the errors considered are zero mean, and hence, the nonlinear least squares problem becomes

$$\hat{\mathcal{X}} = \arg\min_{\mathcal{X}} \frac{1}{2} \sum_{i=1}^{n} \mathbf{e}_i\left(\mathcal{X}_i\right)^{\mathsf{T}} \boldsymbol{\Sigma}_i^{-1} \mathbf{e}_i\left(\mathcal{X}_i\right), \tag{35}$$

where the dependence of each error term on a measurement $\mathcal{Y}_i$ has been dropped. This is also commonly written as

$$\hat{\mathcal{X}} = \arg\min_{\mathcal{X}} \frac{1}{2} \sum_{i=1}^{n} \left\|\mathbf{e}_i\left(\mathcal{X}_i\right)\right\|^2_{\boldsymbol{\Sigma}_i^{-1}}, \tag{36}$$

where $\left\|\mathbf{e}_i\left(\mathcal{X}_i\right)\right\|^2_{\boldsymbol{\Sigma}_i^{-1}}$ is the *squared Mahalanobis distance*.

1.4.1 Solving Nonlinear Least Squares

Nonlinear least squares problems in the form of (36), rely on a *linearization* of the error terms $\mathbf{e}_i(\mathcal{X})$ with respect to the state $\mathcal{X}_i$. Formally, the individual error Jacobians are defined as

$$\mathbf{H}_i = \left.\frac{D\mathbf{e}_i(\mathcal{X}_i)}{D\mathcal{X}_i}\right|_{\mathcal{X}_i=\bar{\mathcal{X}}_i}, \tag{37}$$

where $\bar{\mathcal{X}}_i$ is the evaluation point of the Jacobian. Stacking all error terms

$$\mathbf{e}(\mathcal{X}) = \begin{bmatrix}\mathbf{e}_1(\mathcal{X}) & \cdots & \mathbf{e}_n(\mathcal{X})\end{bmatrix}, \tag{38}$$

the full error Jacobian can be written as

$$\mathbf{H} = \left.\frac{D\mathbf{e}(\mathcal{X})}{D\mathcal{X}}\right|_{\mathcal{X}=\bar{\mathcal{X}}}. \tag{39}$$

Iterative nonlinear least squares methods such as Gauss-Newton or Levenberg-Marquardt, utilize this error Jacobian to compute an update to the state variables. For example, Gauss-Newton computes a step as

$$\delta\hat{\mathbf{x}} = \left(\mathbf{H}\mathbf{W}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{W}\mathbf{e}, \tag{40}$$

which is then used to update the state estimate as $\hat{\mathcal{X}} \leftarrow \hat{\mathcal{X}} \oplus \delta\hat{\mathbf{x}}$. Note that in the update step, a left-or-right defintion of the $\oplus$ operator may be used, as long as the same definition is used in computing the error Jacobian in (39). As solving nonlinear least squares problems relies on the Jacobian of the error terms, the form of the Jacobians for both a left and right perturbation will be derived for common sensor models in the following section.

2 Sensor Models and Error Terms

2.1 Relative Pose Measurements

In many SLAM problems, it is assumed that a sensor directly measures relative poses. These relative pose measurements could come, for example, from a visual odometry or LiDAR odometry pipeline. Relative pose measurements form the basis of the task of *pose graph optimization*, where the task is to estimate the poses of a vehicle relative to a base frame, given relative pose measurements between an arbitrary set of poses.

Denote robot poses at times $t = t_i$ and $t = t_j$ as $\mathbf{T}_i, \mathbf{T}_j \in SE(3)$. The pose at time $t = t_k$ is of the form

$$\mathbf{T}_k = \begin{bmatrix}\mathbf{C}_{ab_k} & \mathbf{r}_a^{z_k w} \\ \mathbf{0} & 1\end{bmatrix}, \tag{41}$$

where $\mathbf{C}_{ab_i} \in SO(3)$ is the orientation of the robot at time t_i with respect to the base frame $\mathcal{F}_a$, and $\mathbf{r}_a^{zw}$ is the robot position resolved in an arbitrary base reference frame. The true relative pose of the robot between times $t = t_i$ and $t = t_j$ is given by

$$\begin{aligned}\mathbf{T}_{ij} &= \mathbf{T}_i^{-1}\mathbf{T}_j &(42)\\ &= \begin{bmatrix}\mathbf{C}_{ab_i}^{\mathsf{T}} & -\mathbf{r}_{b_i}^{z_i w}\\ \mathbf{1} & 0\end{bmatrix}\begin{bmatrix}\mathbf{C}_{ab_j} & \mathbf{r}_a^{z_j w}\\ \mathbf{0} & 1\end{bmatrix} &(43)\\ &= \begin{bmatrix}\mathbf{C}_{b_i b_j} & \mathbf{r}_{b_i}^{z_j z_i}\\ \mathbf{0} & 1\end{bmatrix}. &(44)\end{aligned}$$

Noisy measurements of the relative pose are denoted $\tilde{\mathbf{T}}_{ij}$, are then assumed to be of the form

$$\tilde{\mathbf{T}}_{ij} = \mathbf{T}_{ij} \oplus \boldsymbol{\eta}_{ij}, \tag{45}$$

where $\boldsymbol{\eta}_{ij} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{ij})$ represents Gaussian noise with covariance $\boldsymbol{\Sigma}_{ij}$. Note that it is assumed that the noise is modelled in the Lie algebra and is added to the true relative pose using either the “left” or “right” definition of the $\oplus$ operator. For example, for a “left” defintion, the noise is modelled as

$$\tilde{\mathbf{T}}_{ij} = \operatorname{Exp}\left(\boldsymbol{\eta}_{ij}\right)\mathbf{T}_{ij}, \tag{46}$$

where $\operatorname{Exp}(\cdot) : \mathbb{R}^6 \to SE(3)$ is the exponential map of $SE_(3)$.

With these relative pose measurements, an error term in batch estimation can be formed as

$$\begin{aligned}\mathbf{e}_{ij}\left(\mathbf{T}_i, \mathbf{T}_j\right) &= \tilde{\mathbf{T}}_{ij} \ominus \mathbf{T}_{ij}, &(47)\\ &= \tilde{\mathbf{T}}_{ij} \ominus \left(\mathbf{T}_i^{-1}\mathbf{T}_j\right). &(48)\end{aligned}$$

which takes the difference between the true relative pose and the measured relative pose.

To derive the Jacobians of this residual with respect to the state variables $\mathbf{T}_i$ and $\mathbf{T}_j$, the chain rule can be used to write

$$\frac{D\mathbf{e}(\mathbf{T}_i, \mathbf{T}_j)}{D\mathbf{T}_k} = \frac{D\mathbf{e}(\mathbf{T}_i, \mathbf{T}_j)}{D\mathbf{T}_{ij}}\frac{D\mathbf{T}_{ij}}{D\mathbf{T}_k}, \quad k \in \{i, j\}, \tag{49}$$

where the Lie group definition of the Jacobian is used, defined in (16). The first Jacobian in the chain rule is simply the Jacobian of the $\ominus$ operator, while the second Jacobian is the Jacobian of the function $\mathbf{T}_{ij} = \mathbf{T}_i^{-1}\mathbf{T}_j$ with respect to $\mathbf{T}_i$ or $\mathbf{T}_j$.

The following sections will derive these individual Jacobians for both the “left” and “right” definitions of the $\oplus$ and $\ominus$ operators.

2.1.1 Left Perturbation

In the expression for the error Jacobian (49), the first Jacobian is the Jacobian of the error with respect to the relative pose, and is simply the Jacobian of the $\ominus$ operator with respect to the second argument. For a left perturbation, this Jacobian is given by $\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}} = -\mathbf{J}_r(\boldsymbol{\tau})^{-1}$. To derive the second Jacobian, perturn both $\mathbf{T}_{ij}$ and $\mathbf{T}_i$ as

$$
\begin{aligned}
\operatorname{Exp}\left(\delta\boldsymbol{\xi}^{T}\right)\bar{\mathbf{T}}_{ij} &= \left(\operatorname{Exp}(\delta\boldsymbol{\xi}_i)\bar{\mathbf{T}}_i\right)^{-1}\bar{\mathbf{T}}_j && (50)\\
&= \bar{\mathbf{T}}_i^{-1}\operatorname{Exp}(-\delta\boldsymbol{\xi}_i)\bar{\mathbf{T}}_j && (51)\\
&= \bar{\mathbf{T}}_i^{-1}\operatorname{Exp}(-\delta\boldsymbol{\xi}_i)\bar{\mathbf{T}}_i\bar{\mathbf{T}}_i^{-1}\bar{\mathbf{T}}_j && (52)\\
&= \exp\left(-\left(\operatorname{Ad}\left(\bar{\mathbf{T}}_i^{-1}\right)\delta\boldsymbol{\xi}_i\right)^{\wedge}\right)\bar{\mathbf{T}}_{ij} && (53)\\
\operatorname{Exp}\left(\delta\boldsymbol{\xi}^{T_{ij}}\right) &= \exp\left(-\left(\operatorname{Ad}\left(\bar{\mathbf{T}}_i^{-1}\right)\delta\boldsymbol{\xi}_i\right)^{\wedge}\right) && (54)\\
\delta\boldsymbol{\xi}^{T_{ij}} &= -\operatorname{Ad}\left(\bar{\mathbf{T}}_i^{-1}\right)\delta\boldsymbol{\xi}_i. && (55)
\end{aligned}
$$

Next, the Jacobian of the errorwith respect to $\mathbf{T}_j$ can be found in a similar manner as

$$
\frac{D\mathbf{e}}{D\mathbf{T}_j} = \frac{D\mathbf{e}}{D\mathbf{T}_{ij}}\frac{D\mathbf{T}_{ij}}{D\mathbf{T}_j}. \tag{56}
$$

The first Jacobian is the same as before, and the second Jacobian can be found by perturbing both sides as

$$
\begin{aligned}
\operatorname{Exp}\left(\delta\boldsymbol{\xi}^{T_{ij}}\right)\bar{\mathbf{T}}_{ij} &= \bar{\mathbf{T}}_i^{-1}\operatorname{Exp}\left(\delta\boldsymbol{\xi}_j\right)\bar{\mathbf{T}}_j && (57)\\
&= \exp\left(\left(\operatorname{Ad}\left(\bar{\mathbf{T}}_i^{-1}\right)\delta\boldsymbol{\xi}_j\right)^{\wedge}\right)\bar{\mathbf{T}}_{ij} && (58)\\
\delta\boldsymbol{\xi}^{T_{ij}} &= \operatorname{Ad}\left(\bar{\mathbf{T}}_i^{-1}\right)\delta\boldsymbol{\xi}_j. && (59)
\end{aligned}
$$

Finally, we require the statistic on the noise on the residual, $\mathbf{e}_{ij}$, given that the noise on the relative pose measurement is Gaussian. The residual including the noise is written as

$$
\mathbf{e}_{ij} = \left(\bar{\mathbf{T}}_{ij}\oplus\boldsymbol{\eta}_{ij}\right)\ominus\bar{\mathbf{T}}_{ij}. \tag{60}
$$

This is an instance of passing a Gaussian through a nonlinearity, and to determine the statistics on the output, we can linearize this expression with respect to $\boldsymbol{\eta}_{ij}$ as

$$
\frac{D\mathbf{e}_{ij}}{D\boldsymbol{\eta}_{ij}} = \frac{D\mathbf{e}_{ij}}{D\tilde{\mathbf{T}}_{ij}}\frac{D\tilde{\mathbf{T}}_{ij}}{D\boldsymbol{\eta}_{ij}}. \tag{61}
$$

To solve for these two Jacobians, we require the Jacobians of the $\oplus$ and $\ominus$ operators, given by

$$
\begin{aligned}
\mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\ominus\mathcal{X}} &= \mathbf{J}_{\ell}\left(\boldsymbol{\tau}\right)^{-1} && (62)\\
\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}} &= \mathbf{J}_{\ell}\left(\boldsymbol{\tau}\right), && (63)
\end{aligned}
$$

and hence, this Jacobian is simply identity.

This can also be found by directly examining how the noise enters the error, as

$$
\mathbf{e}_{ij} = \operatorname{Log}\left(\left(\operatorname{Exp}\left(\boldsymbol{\eta}_{ij}\right)\bar{\mathbf{T}}_{ij}\right)\bar{\mathbf{T}}_{ij}^{-1}\right), \tag{64}
$$

where we see that $\mathbf{e}_{ij}\sim\mathcal{N}\left(\mathbf{0},\boldsymbol{\Sigma}_{ij}\right)$.

2.1.2 Right Perturbation

For a "right" perturbation of the state, the error is defined as

$$
\begin{aligned}
\mathbf{e}_{ij} &= \operatorname{Log}\left(\tilde{\mathbf{T}}_{ij} \ominus \mathbf{T}_{ij}\right) && (65)\\
&= \operatorname{Log}\left(\left(\mathbf{T}_{ij}^{-1}\tilde{\mathbf{T}}_{ij}\right)\right) && (66)\\
&= \operatorname{Log}\left(\left(\mathbf{T}_{i}^{-1}\mathbf{T}_{j}\right)^{-1}\tilde{\mathbf{T}}_{i}j\right), && (67)\\
&= \operatorname{Log}\left(\mathbf{T}_{j}^{-1}\mathbf{T}_{i}\tilde{\mathbf{T}}_{ij}\right). && (68)
\end{aligned}
$$

The Jacobian of this residual with respect to the state variables can now be derived. The Jacobians with respect to $\mathbf{T}_i$ are found as

$$
\frac{D\mathbf{e}}{D\mathbf{T}_i} = \frac{D\mathbf{e}}{D\mathbf{T}_{ij}}\frac{D\mathbf{T}_{ij}}{D\mathbf{T}_i}. \tag{69}
$$

The first Jacobian is the Jacobian of the error with repsect to the relative pose, and is the Jacobian of the $\ominus$ operator with respect to the second argument. For a right perturbation, this Jacobian is given by $\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}} = -\mathbf{J}_{\ell}\left(\boldsymbol{\tau}\right)^{-1}$. To derive the second Jacobian, perturb both $\mathbf{T}_{ij}$ and $\mathbf{T}_i$ as

$$
\begin{aligned}
\bar{\mathbf{T}}_{ij}\operatorname{Exp}\left(\delta\boldsymbol{\xi}^{T_{ij}}\right) &= \left(\bar{\mathbf{T}}_i\operatorname{Exp}\left(\delta\boldsymbol{\xi}_i\right)\right)^{-1}\bar{\mathbf{T}}_j && (70)\\
&= \operatorname{Exp}\left(-\delta\boldsymbol{\xi}_i\right)\bar{\mathbf{T}}_i^{-1}\bar{\mathbf{T}}_j && (71)\\
&= \bar{\mathbf{T}}_{ij}\bar{\mathbf{T}}_{ij}^{-1}\operatorname{Exp}\left(-\delta\boldsymbol{\xi}_i\right)\bar{\mathbf{T}}_{ij}, && (72)\\
&= \bar{\mathbf{T}}_{ij}\exp\left(-\left(\operatorname{Ad}(\bar{\mathbf{T}}_{ij}^{-1})\delta\boldsymbol{\xi}_i\right)^{\wedge}\right) && (73)\\
\operatorname{Exp}\left(\delta\boldsymbol{\xi}^{T_{ij}}\right) &= \exp\left(-\left(\operatorname{Ad}(\bar{\mathbf{T}}_{ij}^{-1})\delta\boldsymbol{\xi}_i\right)^{\wedge}\right), && (74)\\
\delta\boldsymbol{\xi}^{T_{ij}} &= -\operatorname{Ad}\left(\bar{\mathbf{T}}_{ij}\right)\delta\boldsymbol{\xi}_i. && (75)
\end{aligned}
$$

Next, the Jacobian of the error with respect to $\mathbf{T}_j$ can be found in a similar manner. Perturbing both sides of the relative pose measuement yields

$$
\begin{aligned}
\bar{\mathbf{T}}_{ij}\operatorname{Exp}\left(\delta\boldsymbol{\xi}^{T_{i}j}\right) &= \bar{\mathbf{T}}_i^{-1}\bar{\mathbf{T}}_j\operatorname{Exp}\left(\delta\boldsymbol{\xi}_j\right), && (76)\\
&= \bar{\mathbf{T}}_{ij}\operatorname{Exp}\left(\delta\boldsymbol{\xi}_j\right) && (77)\\
\delta\boldsymbol{\xi}^{T_{ij}} &= \delta\boldsymbol{\xi}_j. && (78)
\end{aligned}
$$

2.2 Relative Landmark Position Measurments

In many SLAM problems, it is assumed that a sensor directly measures a point landmark in 3D space, but resolved in the frame of the sensor. Denote the global frame as $\mathcal{F}_a$, the position of the point landmark resolved in $\mathcal{F}_a$ as $\mathbf{r}_a^{pw}$, and the position of the robot resolved in the global frame as $\mathbf{r}_a^{zw}$. Additionally, denote the robot body frame as $\mathcal{F}_b$, as assume that the body frame coincides with the sensor frame. The direction-cosine-matrix DCM relating the attitude of the robot frame to the attitude of the global frame is denoted $\mathbf{C}_{ab} \in SO(3)$. The robot attitude and position can be Noisy measurements of the landmark resolved in the body frame are given by

$$
\mathbf{y}_k = \mathbf{g}_k\left(\mathbf{T}_{ab}, \mathbf{v}_k\right) + \mathbf{v}_k = \mathbf{C}_{ab}^{\mathsf{T}}\left(\mathbf{r}_a^{pw} - \mathbf{r}_a^{zw}\right) + \mathbf{v}_k, \tag{79}
$$

where $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$ is Gaussian noise. In SLAM, we are interested in estimating both the position of the landmark, $\mathbf{r}_a^{pw}$, in addition to the the pose of the robot, $\mathbf{T}_{ab}$ in $SE(3)$. Thus, the Jacobians of the measurement model with respect to both the robot pose and the landmark are required. These Jacobians are respectively written as

$$\frac{D\mathbf{g}_k(\mathbf{T}_{ab}, \mathbf{r}_a^{pw})}{D\mathbf{T}_{ab}}, \quad \frac{D\mathbf{g}_k(\mathbf{T}_{ab}, \mathbf{r}_a^{pw})}{D\mathbf{r}_a^{pw}}. \tag{80}$$

To start, the Jacobian of the measurement model with respect to the pose will be derived for both a left and a right perturbation of the state.

2.2.1 Left Perturbation Jacobians

To start, consider the perturbation scheme on the pose given by

$$\mathbf{T}_{ab} = \operatorname{Exp}(\delta\boldsymbol{\xi})\,\bar{\mathbf{T}}_{ab}, \tag{81}$$

$$\begin{bmatrix} \mathbf{C}_{ab} & \mathbf{r}_a^{zw} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \delta\mathbf{C} & \delta\mathbf{r} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{C}}_{ab} & \bar{\mathbf{r}}_a^{zw} \\ \mathbf{0} & 1 \end{bmatrix} \tag{82}$$

$$= \begin{bmatrix} \delta\mathbf{C}\bar{\mathbf{C}}_{ab} & \delta\mathbf{C}\bar{\mathbf{r}}_a^{zw} + \delta\mathbf{r} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{83}$$

where $\delta\mathbf{C} = \operatorname{Exp}\left(\delta\boldsymbol{\xi}^{\phi}\right)$ and $\delta\mathbf{r} = \mathbf{J}_l\delta\boldsymbol{\xi}^{\mathrm{r}}$. Neglecting subscripts, we thus have the following individual perturbations for the robot state.

$$\mathbf{C} = \delta\mathbf{C}\bar{\mathbf{C}}, \tag{84}$$

$$\mathbf{r} = \delta\mathbf{C}\bar{\mathbf{r}} + \delta\mathbf{r}. \tag{85}$$

Perturbing both sides of the measurement model using this perturbation scheme yields

$$\bar{\mathbf{g}}_k + \delta\mathbf{g}_k = \left(\delta\mathbf{C}\bar{\mathbf{C}}\right)^{\mathsf{T}} \left(\bar{\mathbf{r}}_a^{pw} - \delta\mathbf{C}\bar{\mathbf{r}} - \delta\mathbf{r}\right), \tag{86}$$

$$= \bar{\mathbf{C}}^{\mathsf{T}}\delta\mathbf{C}^{\mathsf{T}} \left(\bar{\mathbf{r}}_a^{pw} - \delta\mathbf{C}\bar{\mathbf{r}} - \delta\mathbf{r}\right), \tag{87}$$

$$= \bar{\mathbf{C}}^{\mathsf{T}} \left(\delta\mathbf{C}^{\mathsf{T}}\bar{\mathbf{r}}_a^{pw} - \bar{\mathbf{r}} - \delta\mathbf{C}^{\mathsf{T}}\delta\mathbf{r}\right). \tag{88}$$

Linearizing by letting $\delta\mathbf{C} \approx \mathbf{1} + \delta\boldsymbol{\xi}^{\phi^\times}$, and $\delta\mathbf{r} \approx \delta\boldsymbol{\xi}^{\mathrm{r}}$, we have

$$\bar{\mathbf{g}}_k + \delta\mathbf{g}_k \approx \bar{\mathbf{C}}^{\mathsf{T}} \left(\left(\mathbf{1} - \delta\boldsymbol{\xi}^{\phi^\times}\right)\bar{\mathbf{r}}_a^{pw} - \bar{\mathbf{r}} - \left(\mathbf{1} - \delta\boldsymbol{\xi}^{\phi^\times}\right)\delta\boldsymbol{\xi}^{\mathrm{r}}\right). \tag{89}$$

Neglecting higher order terms and subtracting the nominal solution yields

$$\delta\mathbf{g}_k \approx \bar{\mathbf{C}}^{\mathsf{T}} \left(-\delta\boldsymbol{\xi}^{\phi^\times}\bar{\mathbf{r}}_a^{pw} - \delta\boldsymbol{\xi}^{\mathrm{r}}\right), \tag{90}$$

$$= \bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{r}}_a^{pw^\times}\delta\boldsymbol{\xi}^{\phi} - \bar{\mathbf{C}}^{\mathsf{T}}\delta\boldsymbol{\xi}^{\mathrm{r}}. \tag{91}$$

Hence, the Jacobian of the measurement model with respect to the pose is given by

$$\frac{D\mathbf{g}_k}{D\mathbf{T}_{ab}} = \begin{bmatrix} \bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{r}}_a^{pw^\times} & -\bar{\mathbf{C}}^{\mathsf{T}} \end{bmatrix}. \tag{92}$$

2.2.2 Right Perturbation Jacobians

For a right perturbation of the state, the perturbation scheme is given by

$$\mathbf{T}_{ab} = \bar{\mathbf{T}}_{ab}\,\mathrm{Exp}\left(\delta\boldsymbol{\xi}\right), \tag{93}$$

$$\begin{bmatrix}\mathbf{C}_{ab} & \mathbf{r}_a^{zw}\\ \mathbf{0} & 1\end{bmatrix} = \begin{bmatrix}\bar{\mathbf{C}}_{ab} & \bar{\mathbf{r}}_a^{zw}\\ \mathbf{0} & 1\end{bmatrix}\begin{bmatrix}\delta\mathbf{C} & \delta\mathbf{r}\\ \mathbf{0} & 1\end{bmatrix} \tag{94}$$

$$= \begin{bmatrix}\bar{\mathbf{C}}_{ab}\delta\mathbf{C} & \bar{\mathbf{C}}_{ab}\delta\mathbf{r} + \bar{\mathbf{r}}_a^{zw}\\ \mathbf{0} & 1\end{bmatrix} \tag{95}$$

Neglecting subscripts, the individual perturbations for the robot state are given by

$$\mathbf{C} = \bar{\mathbf{C}}\delta\mathbf{C}, \tag{96}$$

$$\mathbf{r} = \bar{\mathbf{C}}\delta\mathbf{r} + \bar{\mathbf{r}}. \tag{97}$$

Perturbing both sides of the measurement model using this perturbation scheme yields

$$\bar{\mathbf{g}}_k + \delta\mathbf{g}_k = \left(\bar{\mathbf{C}}\delta\mathbf{C}\right)^{\mathsf{T}}\left(\bar{\mathbf{r}}_a^{pw} - \bar{\mathbf{C}}\delta\mathbf{r} - \bar{\mathbf{r}}\right) \tag{98}$$

$$= \delta\mathbf{C}^{\mathsf{T}}\bar{\mathbf{C}}^{\mathsf{T}}\left(\bar{\mathbf{r}}_a^{pw} - \bar{\mathbf{C}}\delta\mathbf{r} - \bar{\mathbf{r}}\right) \tag{99}$$

$$= \delta\mathbf{C}^{\mathsf{T}}\left(\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{r}}_a^{pw} - \delta\mathbf{r} - \bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{r}}\right) \tag{100}$$

$$= \delta\mathbf{C}^{\mathsf{T}}\left(\bar{\mathbf{g}}_k - \delta\mathbf{r}\right) \tag{101}$$

$$\approx \left(\mathbf{1} - \delta\boldsymbol{\xi}^{\phi^\times}\right)\left(\bar{\mathbf{g}}_k - \delta\boldsymbol{\xi}^{\mathrm{r}}\right), \tag{102}$$

$$\delta\mathbf{g} \approx \bar{\mathbf{g}}_k^{\times}\delta\boldsymbol{\xi}^{\phi} - \delta\boldsymbol{\xi}^{\mathrm{r}}, \tag{103}$$

where $\bar{\mathbf{g}}_k = \bar{\mathbf{C}}_{ab}^{\mathsf{T}}\left(\bar{\mathbf{r}}_a^{pw} - \bar{\mathbf{r}}_a^{zw}\right)$. Hence, the Jacobian of the measurement model with respect to the pose utilizing a right perturbation scheme is given by

$$\frac{D\mathbf{g}_k}{D\mathbf{T}_{ab}} = \begin{bmatrix}\bar{\mathbf{g}}_k^{\times} & -\mathbf{1}\end{bmatrix}. \tag{104}$$

2.2.3 Jacobian with Respect to the Landmark Position

Next, for SLAM problems, we typically require the Jacobian of the measurement model with respect to the landmark position. This Jacobian can be found by perturbing the landmark position as $\mathbf{r}_a^{pw} = \bar{\mathbf{r}}_a^{pw} + \delta\mathbf{r}_a^{pw}$, to yield

$$\bar{\mathbf{g}}_k + \delta\mathbf{g}_k = \bar{\mathbf{C}}^{\mathsf{T}}\left(\bar{\mathbf{r}}_a^{pw} + \delta\mathbf{r}_a^{pw} - \bar{\mathbf{r}}\right), \tag{105}$$

$$\delta\mathbf{g}_k = \bar{\mathbf{C}}^{\mathsf{T}}\delta\mathbf{r}_a^{pw}, \tag{106}$$

and hence the Jacobian is given by

$$\frac{D\mathbf{g}_k(\mathbf{T}_{ab}, \mathbf{r}_a^{pw})}{D\mathbf{r}_a^{pw}} = \bar{\mathbf{C}}_{ab}^{\mathsf{T}}. \tag{107}$$

References

[1] J. Sola, J. Deray, and D. Atchuthan, “A Micro Lie Theory for State Estimation in Robotics,” *arXiv preprint arXiv:1812.01537*, 2018.